# Causal Theory of Dissipative Hydrodynamics

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# Introduction

Ideal hydrodynamics works well in RHIC experiments.
But we still have many questions.
(Kodama,Koide,Denicol&Mota,hep-ph0606161)

- Early thermalization
- Freeze-out procedure
- Fluctuations
- Dissipation

Relativistic dissipative hydrodynamics is not obvious at all!!

The Navier-Stokes equation in the covariant form (Landau-Lifshitz theory&Eckart theory)

- Causality
- Extension of causal theory is not unique.

We propose a new theory of relativistic dissipative hydrodynamics.

# Diffusion equation

Equation of continuity

$$\frac{d}{dt}n = \nabla J$$

Phenomenological law

$$J = D\nabla n$$

Diffusion Equation

$$\frac{d}{dt}n = D\nabla^2 n$$

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However, the propagation speed of signal exceeds the speed of light!

# Memory effect

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Koide,Krein&Ramos, PLB, 636, 96 (2006.)

To solve the problem, we take the time delay effect into account.

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$$J = D \int_{-\infty}^{t} ds \, G(t-s) \nabla n(s)$$

$$J = \frac{D}{\tau_R} \int_{-\infty}^{t} ds \, e^{-(t-s)/\tau_R} \nabla n(s)$$

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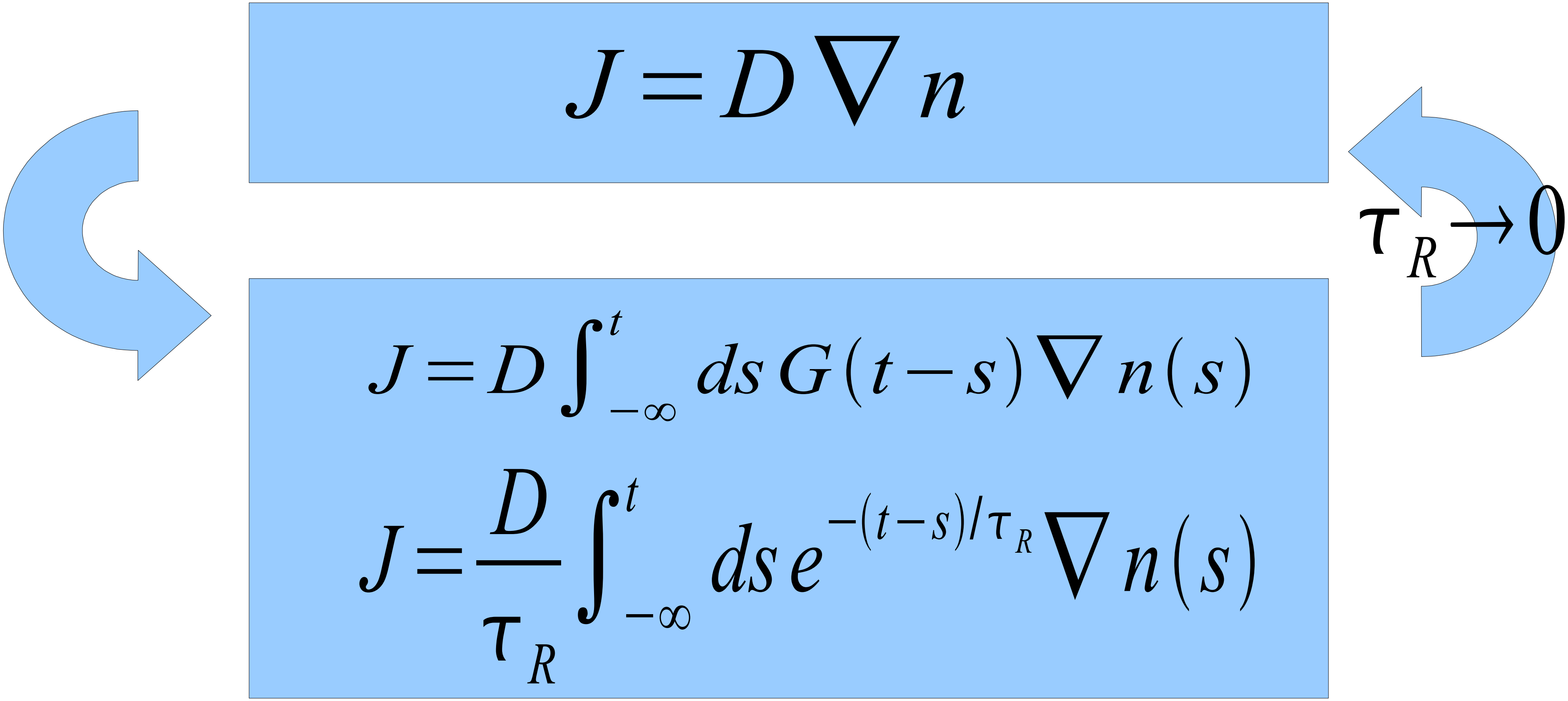

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$$\tau_R \to 0$$

# Causal Diffusion Equation

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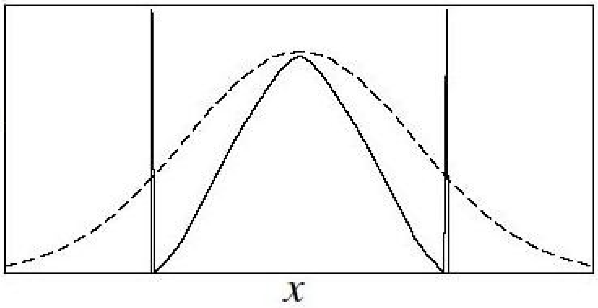

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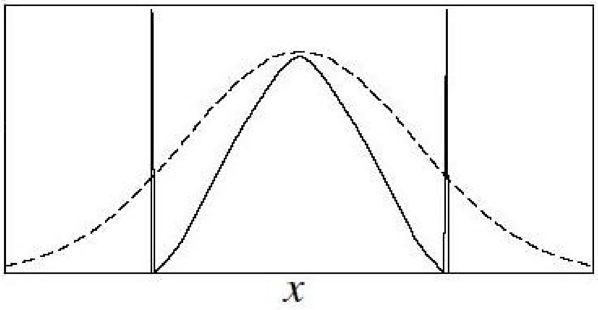

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Propagation speed

$$V = \left( \frac{D}{\tau_R} \right)^{1/2}$$

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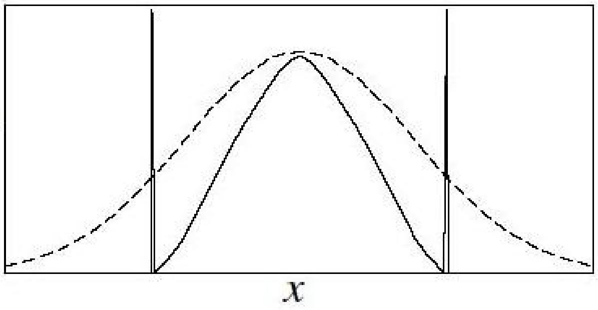

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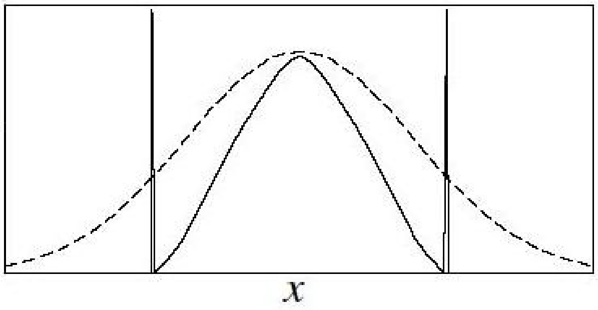

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Propagation speed

$$V = \left( \frac{D}{\tau_R} \right)^{1/2} \longrightarrow \infty$$

# Remarks

- The diffusion equation is not applicable to relativistic phenomena because of causality.

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$$D \underset{\text{F.T.}}{\sim} \langle [ J(r,t), J(0,0) ] \rangle$$

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# Relativistic Dissipative hydrodynamics

## First order theory

The first order theory is the relativistic Navier-Stokes equation.

There is two different choice of the velocity.

- Landau-Lifshitz theory (the velocity is parallel to the energy flow)
- Eckert theory (the velocity is parallel to the particle flux)

The relaxation mechanism is same as that of the diffusion equation.
Thus, the first order theory breaks causality.

# Second order theory

The second order theory is considered to be consistent with causality.
However, the extension is not unique.

- Israel-Stewart theory

  Müller (1967), Israel (1976),Stewart (1977), Muronga (2002,2004),
  Heinz,Song&Chaudhur (2006), Baier,Romatschke&Wiedemann (2006)

- the divergence type theory

  Liu, Müller&Ruggeri (1986), Geroch,Lindblem (1990)

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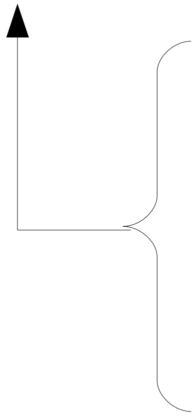

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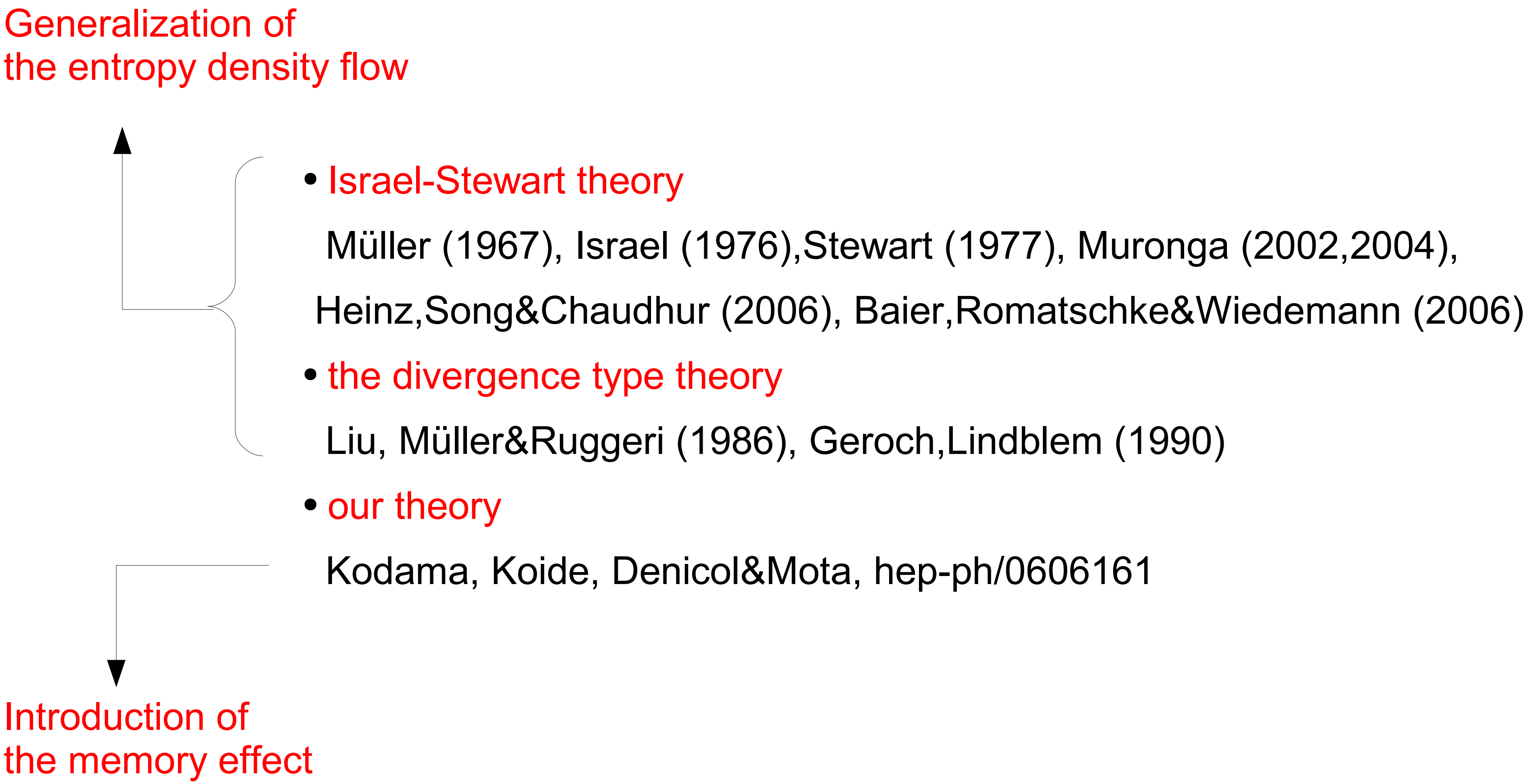

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Introduction of the memory effect

# Landau-Lifshitz theory(I)

Equation of continuity

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu N^\mu = 0$$

Energy-momentum tensor

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P^{\mu\nu}(p + \Pi) + \pi^{\mu\nu}$$

Baryon number current

$$N^\mu = n u^\mu + \nu^\mu$$

$$P^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

# Landau-Lifshitz theory(II)

Entropy density flow

$$\partial_\mu \sigma^\mu = \frac{1}{T}\left(-P^{\mu\nu}\Pi + \pi^{\mu\nu}\right)\partial_\mu u_\nu - \nu^\mu \partial_\mu \frac{\mu}{T} \geqslant 0$$

Second law of thermodynamics

$$\Pi = -\zeta \partial_\mu u^\mu$$

$$P^{\mu\alpha\nu\beta} = \frac{1}{2}\left(P^{\mu\alpha}P^{\nu\beta} + P^{\nu\alpha}P^{\mu\beta}\right) - \frac{1}{3}P^{\mu\nu}P^{\alpha\beta}$$

$$\pi^{\mu\nu} = \eta P^{\mu\alpha\nu\beta}\partial_\alpha u_\beta$$

$$\nu^\mu = -\kappa P^{\mu\nu}\partial_\nu \frac{\mu}{T}$$

# Introduction of Memory effect

$$\Pi = -\int^{\tau} ds\, G(\tau, s)\, \zeta\, \partial_\mu u^\mu$$

$$\pi^{\mu\nu} = P^{\mu\alpha\nu\beta} \int^{\tau} ds\, G(\tau, s)\, \eta\, \partial_\alpha u_\beta$$

$$\nu^\mu = -P^{\mu\nu} \int^{\tau} ds\, G(\tau, s)\, \kappa\, \partial_\nu \frac{\mu}{T}$$

$\tau$ : Local proper time

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Exponential form

$\tau$ : Local proper time

Our main results!

## Our theory

$$\Pi = -\zeta \partial_\mu u^\mu + \tau_R \partial_\tau \Pi$$

$$\tilde{\pi}^{\mu\nu} = -\eta \partial^\mu u^\nu + \tau_R \partial_\tau \tilde{\pi}^{\mu\nu} \qquad \pi^{\mu\nu} = P^{\mu\alpha\nu\beta} \tilde{\pi}_{\alpha\beta}$$

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## Israel-Stewart

$$\Pi = -\zeta \left( \partial_\mu u^\mu - \beta_0 \partial_\tau \Pi - \alpha_0 \partial_\mu \nu^\mu \right) / 3$$

$$\pi^{\mu\nu} = -2\eta P^{\mu\alpha\nu\beta} \left( \partial_\alpha u_\beta - \beta_2 \partial_\tau \pi_{\alpha\beta} - \alpha_1 \partial_\alpha \nu_\beta \right)$$

$$\nu^\mu = -\kappa P^{\mu\nu} \left( \frac{nT}{\epsilon + P} \partial_\nu \frac{\mu}{T} - \beta_1 \partial_\tau \nu_\nu + \alpha_0 \partial_\nu \Pi + \alpha_1 \partial_\alpha \pi^\alpha_\nu \right)$$

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# Entropy production(I)

$$\partial_\mu \sigma^\mu = \frac{1}{T}\left(-P^{\mu\nu}\Pi + \pi^{\mu\nu}\right)\partial_\mu u_\nu - \nu^\mu \partial_\mu \frac{\mu}{T} \geqslant 0$$

This is not satisfied algebraically!

The first term (bulk viscosity)

$$\Pi\,\partial^\mu u_\mu \propto \partial_\mu u^\mu(\tau)\underline{\frac{1}{\tau_R}\int^\tau d\tau' e^{-(\tau-\tau')/\tau_R}\partial_\mu u^\mu(\tau')}$$

Expand near $\tau$

# Entropy production(II)

$$\Pi \partial^{\mu} u_{\mu} \propto \partial_{\mu} u^{\mu} \left[ \partial_{\mu} u^{\mu} - \tau_R \frac{d(\partial_{\mu} u^{\mu})}{d\tau} + O(\tau_R^2) \right]$$

Thus, as far as $\left| \tau_R \frac{d(\partial_{\mu} u^{\mu})}{d\tau} \right| < \left| \partial_{\mu} u^{\mu} \right|$ , entropy production is positive.

- Entropy production is positive only when the relaxation time is enough short and the variation of the system is not too violent.

(For example, the Bjorken scaling solution)

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- Even for the IS, to satisfy the positiveness, we have to extend the concept of the thermodynamic relations.

# Bjorken scaling solution

- Longitudinal expansion
- Physical quantities are functions of the proper time.

$$x^{\mu} = (\tau \sinh y, 0, 0, \tau \cosh y)$$

$$u^{\mu} = (\sinh y, 0, 0, \cosh y)$$

# Second Law of Thermodynamics

$$\partial_\mu \sigma^\mu = \frac{1}{T}\left(1+\frac{2\eta}{3\zeta}\right)\frac{1}{\tau}\frac{\eta}{\gamma}\int_{\tau_0}^{\tau} ds\, e^{-(\tau-s)/\gamma}\frac{1}{s} \geq 0$$

Under the Bjorken scaling solution,
the second law of thermodynamics is still satisfied in our theory.

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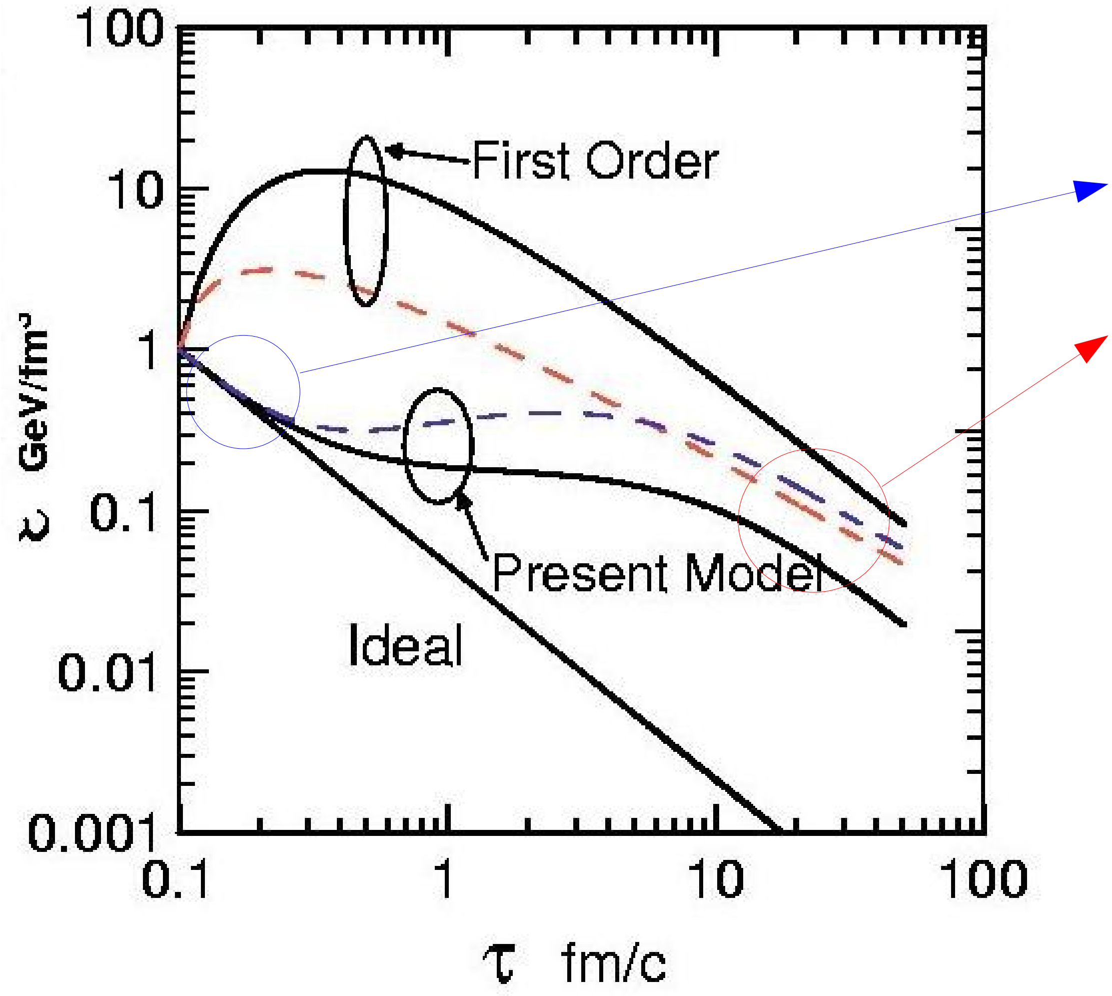
⇕

The scaling solution have no acceleration.
Thus, the time evolution is not too violent.

# Time evolution of energy density



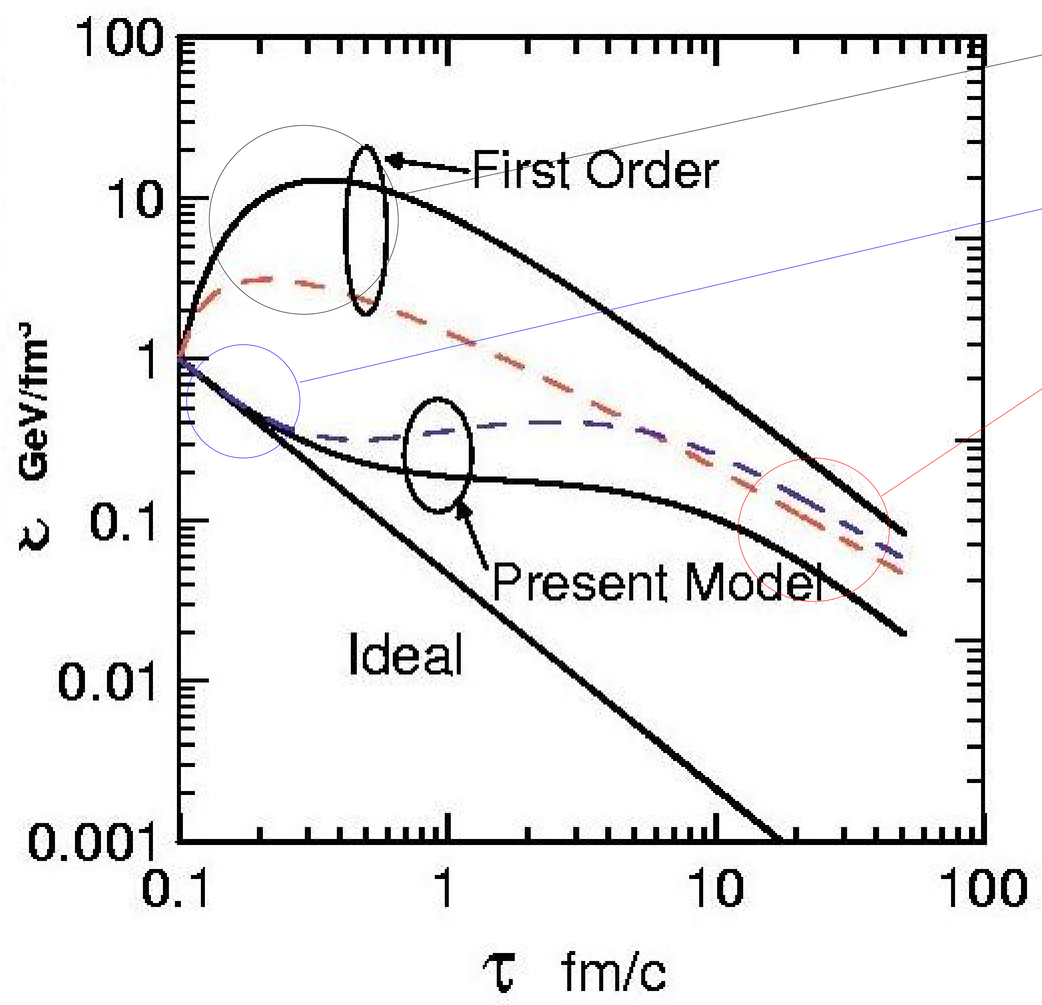
- The initial behavior coincides with the ideal fluid.
- At the late time which is larger than the relaxation time, the behavior is similar to the first order theory.

Same result as Baier et al.
(Baier,Romatschke,Wiedemann, PRC73,064903(2006))

— — — Constant viscosity

——— Temperature dependent viscosity

# Time evolution of energy density



Initial heat-up

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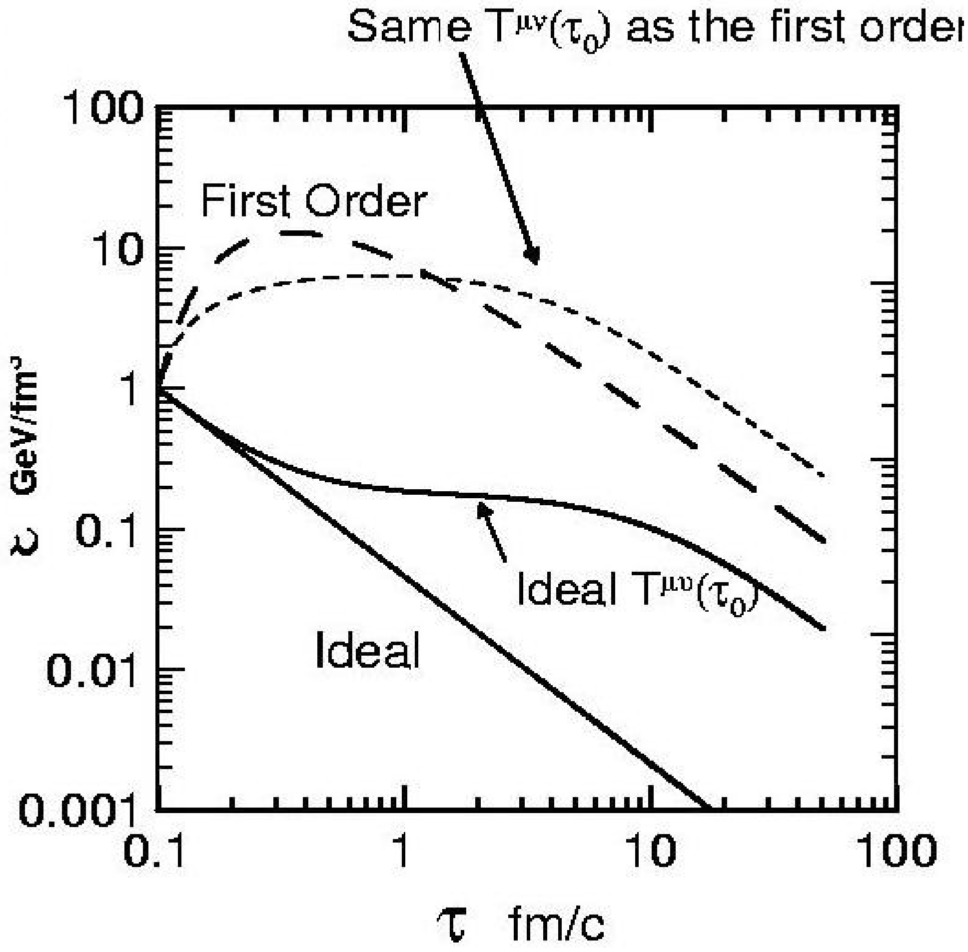
# Initial heat-up

The initial heat-up is not the problem of the first order theory.
At the special choice of the initial conditions, we have the heat-up even in the second order theory.



This is the problem of the scaling solution itself.

# Summary and Outlook

- Relativistic Navier-Stokes equation cannot satisfy causality.
- We introduced the memory effect to the LL theory.
- Our theory contains less parameters than the IS theory and should be easy to solve numerically.
- In causal dissipative theories, the concept of the positive entropy production should be extended.
- The temporal behavior is same as the result of Baier et al., because of no acceleration in the scaling solution. In general case, our result is different from their result.
- 3D numerical calculation.
- Critical slowing down, its enhancement near the QCD critical end point, critical opalescence and piston effect.
- Fluctuating hydrodynamics