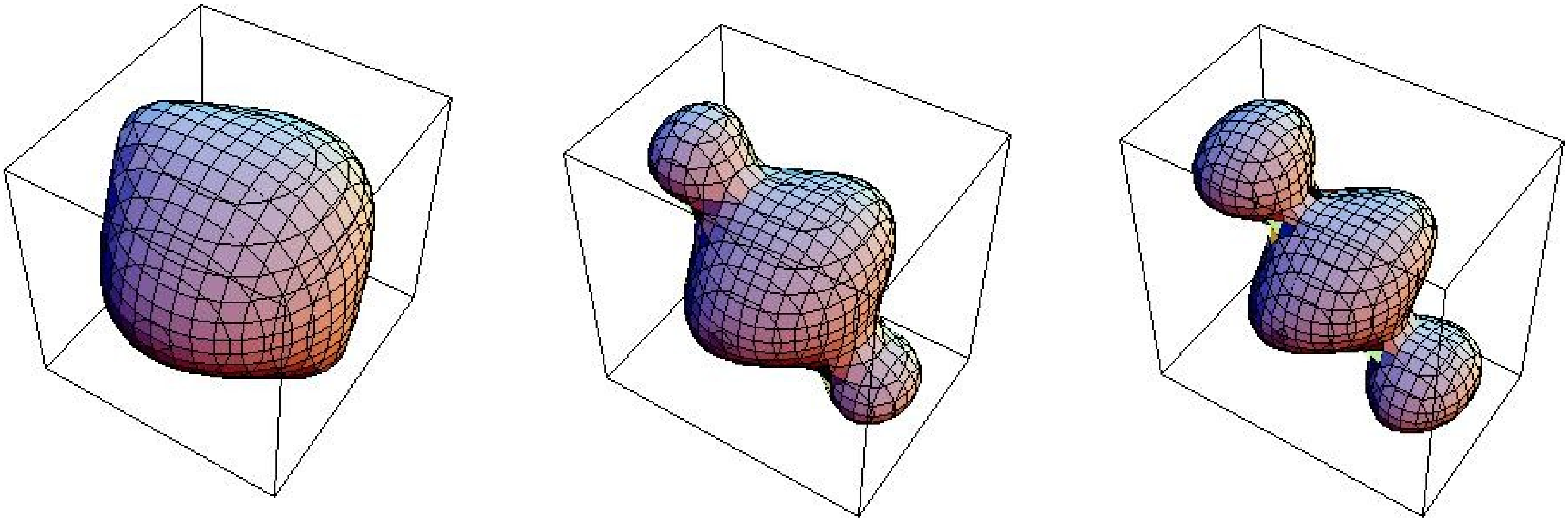
HBT Shape Analysis with q-Cumulants



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Historical: Edgeworth expansion in shape analysis

Csörgő and Hegyi PLB 489,15 (2000) and earlier1993 versions suggested use of Edgeworth expansion

$$
\begin{aligned}
R_2(\boldsymbol{q}) &= \exp[-\textstyle\sum_{ij} R_{ij}^2 q_i q_j] \\
&\cdot \left[1+\frac{1}{6}\kappa_3(q_O)\,H_3(\sqrt{2}\,R_O q_O)+\ldots\right] \\
&\cdot \left[1+\frac{1}{6}\kappa_3(q_S)\,H_3(\sqrt{2}\,R_S q_S)+\ldots\right] \\
&\cdot \left[1+\frac{1}{6}\kappa_3(q_L)\,H_3(\sqrt{2}\,R_L q_L)+\ldots\right]
\end{aligned}
$$

univariate cumulants

Hermite polynomials

Historical: Shape analysis with q-moments

Wiedemann and Heinz PRC 56(1997) suggested q-moments

$$[\![q_i q_j]\!] = \frac{\int d^3 q\, R_2(\boldsymbol{q},\boldsymbol{K})\ \ q_i q_j}{\int d^3 q\, R_2(\boldsymbol{q},\boldsymbol{K})} \qquad i,j = \mathrm{Out}, \mathrm{Side}, \mathrm{Long} \qquad R_2(\boldsymbol{q},\boldsymbol{K}) = C_2(\boldsymbol{q},\boldsymbol{K}) - 1$$

ie defining “probability”

$$f(\boldsymbol{q},\boldsymbol{K}) = \frac{R_2(\boldsymbol{q},\boldsymbol{K})}{\int d^3 q\, R_2(\boldsymbol{q},\boldsymbol{K})}$$

e.g. the 2nd q-moment is

$$[\![q_i q_j]\!] = \int d^3 q\, f(\boldsymbol{q})\ q_i q_j$$

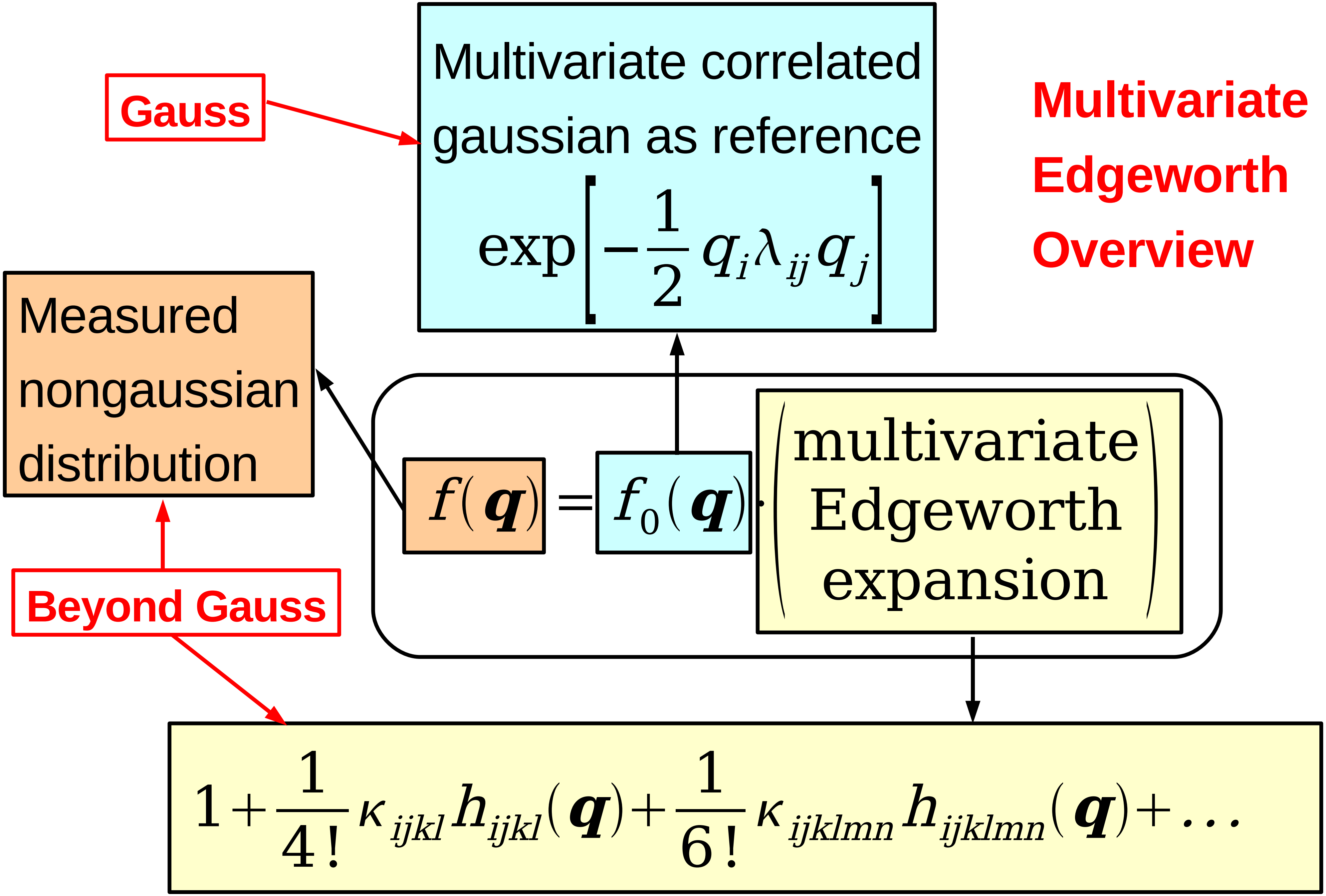
(suppress **K** from now on)

Historical: Discrete multivariate Edgeworth expansion with Poisson reference distribution

$$\begin{aligned} f(\boldsymbol{n}) \;\; &= \;\; \exp\left[\sum_{m=1}^{M} (\kappa_1(m) - \nu_m)(-\nabla_m) \right. \\ &+ \left. \sum_{q=2}^{\infty} \frac{(-1)^q}{q!} \sum_{m_1} \ldots \sum_{m_q} \kappa_q(m_{1,} \ldots, m_q) \nabla_{m_1} \ldots \nabla_{m_q} \right] \mathrm{poisson}(\boldsymbol{n};\boldsymbol{\nu}) \end{aligned}$$

Lipa, Eggers, Buschbeck, hep-ph/9604373, PRD 53,4711(1996)

Multivariate Edgeworth Overview



Gauss

Multivariate correlated gaussian as reference

$$\exp\left[-\frac{1}{2}q_i\lambda_{ij}q_j\right]$$

Measured nongaussian distribution

$$f(\boldsymbol{q})=f_0(\boldsymbol{q})\cdot\left(\begin{array}{c}\text{multivariate}\\ \text{Edgeworth}\\ \text{expansion}\end{array}\right)$$

Beyond Gauss

$$1+\frac{1}{4\,!}\kappa_{ijkl}h_{ijkl}(\boldsymbol{q})+\frac{1}{6\,!}\kappa_{ijklmn}h_{ijklmn}(\boldsymbol{q})+\ldots$$

Multivariate gaussian reference distribution

$$f_o(\boldsymbol{q}) = \frac{1}{(2\pi)^{p/2}\sqrt{det\lambda_{ij}}} \exp\left(-\frac{1}{2}(q_i-\lambda_i)(\lambda^{-1})_{ij}(q_j-\lambda_j)\right)$$

$p =$ dimension (2 = T ,L) or (3 = O ,S ,L) $i, j = 1, \ldots, p$

$q_1 = q_{\mathrm{Out}}$ $q_2 = q_{\mathrm{Side}}$ $q_3 = q_{\mathrm{Long}}$

$\lambda_i = [\![q_i]\!]$ First q-moments

$\lambda_{ij} =$ covariance matrix (2nd order reference cumulants)

$(\lambda^{-1})_{ij} =$ inverse of covariance matrix

Use Einstein summation convention throughout

d = 2 gaussian reference distribution

Covariance matrix (2nd order cumulants)

$$\lambda_{ij} = \begin{pmatrix} \lambda_{TT} & \lambda_{TL} \\ \lambda_{TL} & \lambda_{LL} \end{pmatrix}$$

$$\lambda_{TL} = [\![q_T q_L]\!] - [\![q_T]\!][\![q_L]\!]$$
$$\lambda_{TT} = [\![q_T q_T]\!] - [\![q_T]\!][\![q_T]\!]$$
$$\lambda_{LL} = [\![q_L q_L]\!] - [\![q_L]\!][\![q_L]\!]$$

Standard deviation = sqrt(cumulant)

$$\left\{ \begin{array}{l} \sigma_i = \sqrt{\lambda_{ii}} \\ \sigma_{ij} = \sqrt{\lambda_{ij}} \end{array} \right.$$

Pearson correlation coefficient

$$\left\{ \begin{array}{l} \rho = \dfrac{\sigma_{TL}^2}{\sigma_T \sigma_L} \\ \chi = 1 - \rho^2 \end{array} \right.$$

Inverse cumulant matrix:

$$\lambda_{ij}^{-1} = \begin{pmatrix} \dfrac{1}{\chi \sigma_T^2} & \dfrac{-\rho}{\chi \sigma_T \sigma_L} \\ \dfrac{-\rho}{\chi \sigma_T \sigma_L} & \dfrac{1}{\chi \sigma_L^2} \end{pmatrix} \equiv \begin{pmatrix} 2 R_{TT}^2 & 2 R_{TL}^2 \\ 2 R_{TL}^2 & 2 R_{LL}^2 \end{pmatrix}$$

Note minus sign!

Inverse cumulants are identified with usual “radii”

Gaussian reference distribution for d = 3

Covariance matrix (2nd order cumulants):

$$\lambda_{ij} = \begin{pmatrix} \lambda_{OO} & \lambda_{OS} & \lambda_{OL} \\ \lambda_{OS} & \lambda_{SS} & \lambda_{SL} \\ \lambda_{OL} & \lambda_{SL} & \lambda_{LL} \end{pmatrix}$$

Inverse cumulant matrix:

$$(\lambda^{-1})_{ij} = \frac{1}{det\lambda} \begin{pmatrix} \lambda_{SS}\lambda_{LL} - \lambda_{SL}^2 & \lambda_{OL}\lambda_{SL} - \lambda_{OS}\lambda_{LL} & \lambda_{OS}\lambda_{SL} - \lambda_{OL}\lambda_{SS} \\ \lambda_{OL}\lambda_{SL} - \lambda_{OS}\lambda_{LL} & \lambda_{OO}\lambda_{LL} - \lambda_{OL}^2 & \lambda_{OL}\lambda_{OS} - \lambda_{SL}\lambda_{OO} \\ \lambda_{OS}\lambda_{SL} - \lambda_{OL}\lambda_{SS} & \lambda_{OL}\lambda_{OS} - \lambda_{SL}\lambda_{OO} & \lambda_{OO}\lambda_{SS} - \lambda_{OS}^2 \end{pmatrix}$$

Special case Azimuthal Symmetry:

$$\rho = \sigma_{OL}^2 / (\sigma_O \sigma_L) \qquad \chi = 1 - \rho^2$$

$$(\lambda^{-1})_{ij} = \begin{pmatrix} \frac{1}{\chi \sigma_O^2} & 0 & \frac{-\rho}{\chi \sigma_O \sigma_L} \\ 0 & \frac{1}{\sigma_S^2} & 0 \\ \frac{-\rho}{\chi \sigma_O \sigma_L} & 0 & \frac{1}{\chi \sigma_L^2} \end{pmatrix} \equiv \begin{pmatrix} 2 R_{OO}^2 & 0 & 2 R_{OL}^2 \\ 0 & 2 R_{SS}^2 & 0 \\ 2 R_{OL}^2 & 0 & 2 R_{LL}^2 \end{pmatrix}$$

Edgeworth 1: multivariate notation

each index $i, j, k \ldots$ can run over $(1,2,3) = (O,S,L)$ or (T,L)

	Reference (gaussian)	Measured (nongaussian)
distribution:	$f_0(\boldsymbol{q})$	$f(\boldsymbol{q})$
expectation values:	$E_0[\ ]$	$E[\ ]$
moments:	$\nu_{ijk} = E_0[q_j q_j q_k]$	$\mu_{ijk} = E[q_j q_j q_k]$
cumulants:	λ_{ijk}	κ_{ijk}

e.g.

$$\mu_{112} = E[q_O q_O q_S] = \int d^3 q \ \ f(\boldsymbol{q}) \ q_O q_O q_S$$

Multivariate moments and cumulants

$$\boxed{1}\quad \mu_i = \kappa_i \qquad\qquad \boxed{2}\quad \mu_{ij} = \kappa_{ij} + \kappa_i \kappa_j$$

$$\boxed{3}\quad \mu_{ijk} = \kappa_{ijk} + \overbrace{\kappa_i \kappa_{jk} + \kappa_j \kappa_{ik} + \kappa_k \kappa_{ij}}^{\text{three terms in combinatorics}} + \kappa_i \kappa_j \kappa_k$$

$$= \kappa_{ijk} + \underbrace{\kappa_i \kappa_{jk}[3]}_{\text{combinatorics notation}} + \kappa_i \kappa_j \kappa_k$$

$$\boxed{4}\quad \mu_{ijkl} = \kappa_{ijkl} + \kappa_i \kappa_{jkl}[4] + \kappa_{ij} \kappa_{kl}[3] + \kappa_i \kappa_j \kappa_{kl}[6] + \kappa_i \kappa_j \kappa_k \kappa_l$$

For the special case $\kappa_i = 0 \qquad \kappa_{ij} = 0$

$$\boxed{3}\quad \mu_{ijk} = \kappa_{ijk}$$

$$\boxed{4}\quad \mu_{ijkl} = \kappa_{ijkl}$$

$$\boxed{6}\quad \mu_{ijklmn} = \kappa_{ijklmn} + \kappa_{ijk} \kappa_{lmn}[10]$$

Starting point: multivariate Gram-Charlier series

In terms of **cumulant differences**

$$\eta_i = \kappa_i - \lambda_i$$

$$\eta_{ij} = \kappa_{ij} - \lambda_{ij} \qquad \eta_{ijk} = \overset{\text{measured}}{\kappa_{ijk}} - \overset{\text{reference}}{\lambda_{ijk}} \qquad \text{etc}$$

and **“cumulant difference moments”**

$$\zeta_i = \eta_i = \kappa_i - \lambda_i$$

$$\zeta_{ij} = \eta_{ij} + \eta_i \eta_j = (\kappa_{ij} - \lambda_{ij}) + (\kappa_i - \lambda_i)(\kappa_j - \lambda_j)$$

the **generic Gram-Charlier series is**

$$f(\boldsymbol{q}) = f_0(\boldsymbol{q}) \left[1 + \zeta_i h_i + \frac{1}{2!} \zeta_{ij} h_{ij} + \frac{1}{3!} \zeta_{ijk} h_{ijk} + \frac{1}{4!} \zeta_{ijkl} h_{ijkl} + \frac{1}{5!} \zeta_{ijklm} h_{ijklm} + \ldots \right]$$

with

$$h_i = \frac{1}{f_0} \frac{-\partial f_0}{\partial q_i} \qquad \text{etc}$$

$$h_{ij} = \frac{1}{f_0} \frac{\partial^2 f_0}{\partial q_i \, \partial q_j}$$

Reduction to Edgeworth series with gaussian reference

$$f(\boldsymbol{q}) = f_0(\boldsymbol{q})\left[1+\zeta_i h_i + \frac{1}{2!}\zeta_{ij}h_{ij} + \frac{1}{3!}\zeta_{ijk}h_{ijk} + \frac{1}{4!}\zeta_{ijkl}h_{ijkl} + \frac{1}{5!}\zeta_{ijklm}h_{ijklm} + \ldots\right]$$

1. Construct the reference gaussian to have the same radii as the measured ones:

$$\lambda_i \equiv \kappa_i \qquad \lambda_{ij} \equiv \kappa_{ij}$$

$$\zeta_i = 0 \qquad \zeta_{ij} = 0$$

2. Higher-order cumulants for gaussian are identically zero, and so

$$\begin{aligned}\zeta_{ijk} &= \kappa_{ijk}\\ \zeta_{ijkl} &= \kappa_{ijkl}\\ \zeta_{ijklm} &= \kappa_{ijklm}\\ \zeta_{ijklmn} &= \kappa_{ijklmn} + \kappa_{ijk}\kappa_{lmn}[10]\end{aligned}$$

3. Even q-parity $C_2(-\boldsymbol{q}) = C_2(\boldsymbol{q})$ implies that all odd cumulants vanish:

$\kappa_{ijk} = 0, \quad \kappa_{ijklm} = 0$ and so $\zeta_{ijk} = 0, \quad \zeta_{ijklm} = 0, \quad \zeta_{ijklmn} = \kappa_{ijklmn}$

Resulting Edgeworth expansion (only 4th and 6th order terms survive):

$$f(\boldsymbol{q}) = f_0(\boldsymbol{q})\left[1 + \frac{1}{4!}\kappa_{ijkl}h_{ijkl} + \frac{1}{6!}\kappa_{ijklmn}h_{ijklmn} + \ldots\right]$$

Occupation number formulation

$$f(\boldsymbol{q}) = f_0(\boldsymbol{q})\left[1 + \frac{1}{4!}\kappa_{ijkl}\underset{\substack{\uparrow\\ \boxed{\text{81 terms}}}}{h_{ijkl}} + \frac{1}{6!}\kappa_{ijklmn}\underset{\substack{\uparrow\\ \boxed{\text{729 terms}}}}{h_{ijklmn}} + \ldots\right]$$

For gaussian $f_0(\boldsymbol{q})$ the $h_{ijk\ldots}(\boldsymbol{q})$ are “hermite tensors”

Since hermite tensors are fully symmetric in their indices, we change to **“occupation number notation”:**

$$\begin{aligned} H_{n_1 n_2 n_3} &\equiv h_{ijk\ldots} \\ C_{n_1 n_2 n_3} &\equiv \kappa_{ijk\ldots} \end{aligned}$$

with

n_1 occurrences of q_1

n_2 occurrences of q_2

n_3 occurrences of q_3

e.g.

$$\begin{aligned} H_{400} &= h_{1111} \\ H_{310} &= h_{1112} = h_{1121} = \ldots \\ H_{022} &= h_{2233} = h_{2323} = \ldots \end{aligned}$$

$$\begin{aligned} C_{121} &= \llbracket q_1 q_2^2 q_3 \rrbracket - 2\llbracket q_1 q_2 \rrbracket \llbracket q_2 q_3 \rrbracket - \llbracket q_1 q_3 \rrbracket \llbracket q_2^2 \rrbracket \\ C_{400} &= \llbracket q_1^4 \rrbracket - 3\llbracket q_1^2 \rrbracket \llbracket q_1^2 \rrbracket \end{aligned}$$

Multivariate Edgeworth series: final form

$$\frac{f(\boldsymbol{q})}{f_0(\boldsymbol{q})} = 1+\frac{1}{24}\Big[\ C_{400}H_{400}[3]+4\,C_{310}H_{310}[6] +6\,C_{220}H_{220}[3]+12\,C_{211}H_{211}[3]\ \Big] +\frac{1}{720}\Big[\ C_{600}H_{600}[3]+6\,C_{510}H_{510}[6] +15\,C_{420}H_{420}[6]+30\,C_{411}H_{411}[3] +20\,C_{330}H_{330}[3]+60\,C_{321}H_{321}[6]\ \Big]$$

3d (O,S,L) case

Expansions are dominated by the combinatorics; so mixed-q_i-cumulants are much more important than univariate cumulants.

$$\frac{f(\boldsymbol{q})}{f_0(\boldsymbol{q})} = 1+\frac{1}{24}\Big[\ C_{40}H_{40}[2]+4\,C_{31}H_{31}[2]+6\,C_{22}H_{22}\ \Big] +\frac{1}{720}\Big[\ C_{60}H_{60}[2]+6\,C_{51}H_{51}[2]+15\,C_{42}H_{42}[2]+20\,C_{33}H_{33}\ \Big]$$

2d (T,L) case

Hermite tensors

Notation: define dimensionless variables (scaled by standard deviations)

$$z_1 = \frac{q_1}{\sigma_1} = \frac{q_O}{\sigma_O} \qquad z_2 = \frac{q_2}{\sigma_2} = \frac{q_S}{\sigma_S} \qquad z_3 = \frac{q_3}{\sigma_3} = \frac{q_L}{\sigma_L} = q_L R_L \sqrt{2}$$

Pearson corr. coefficient

1st order hermite tensors: $h_1 = \frac{z_1 - \rho z_3}{\chi \sigma_1} \qquad h_2 = \frac{z_2}{\sigma_2} \qquad h_3 = \frac{z_3 - \rho z_1}{\chi \sigma_3}$

4th order hermite tensors:

(Azimuthally symmetric case!)

$$\begin{aligned}
H_{400} &= h_1^4 - 6 h_1^2 \lambda_{11}^{-1} + 3 (\lambda^{-1})_{11}^2 \\
H_{310} &= h_1^3 h_2 - 3 h_1 h_2 \lambda_{11}^{-1} \\
H_{220} &= h_1^2 h_2^2 - h_1^2 \lambda_{22}^{-1} - h_2^2 \lambda_{11}^{-1} \\
H_{211} &= h_1^2 h_2 h_3 - 2 h_1 h_2 \lambda_{13}^{-1} - h_2 h_3 \lambda_{11}^{-1}
\end{aligned}$$

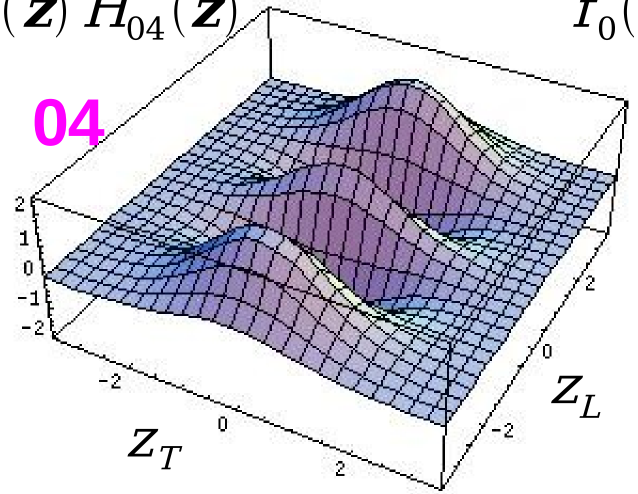
For $\rho = 0$, these factorise into the usual hermite polynomials:

$$H_{400} = \frac{z_1^4 - 6 z_1^2 + 3}{\sigma_1^4} \qquad H_{310} = \left(\frac{z_1^3 - 3 z_1}{\sigma_1^3} \right) \left(\frac{z_2}{\sigma_2} \right) \qquad H_{211} = \frac{H_2(z_1)\, H_1(z_2)\, H_1(z_3)}{\sigma_1^2\, \sigma_2^1\, \sigma_3^1}$$

Hermite tensors 3: two-dimensional (T,L)

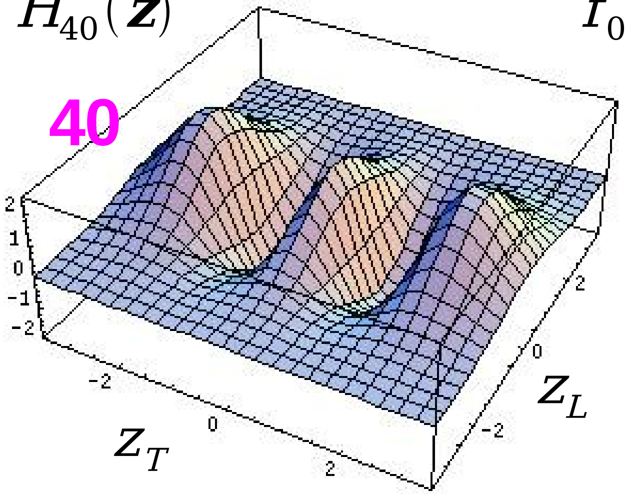
$\boldsymbol{z}=(z_T, z_L)$ $\qquad z_i=\dfrac{q_i}{\sigma_i}$ $\qquad$ ie everything is scaled by gaussian's widths

$f_0(\boldsymbol{z})\, H_{04}(\boldsymbol{z})$



04

$f_0(\boldsymbol{z})\, H_{40}(\boldsymbol{z})$



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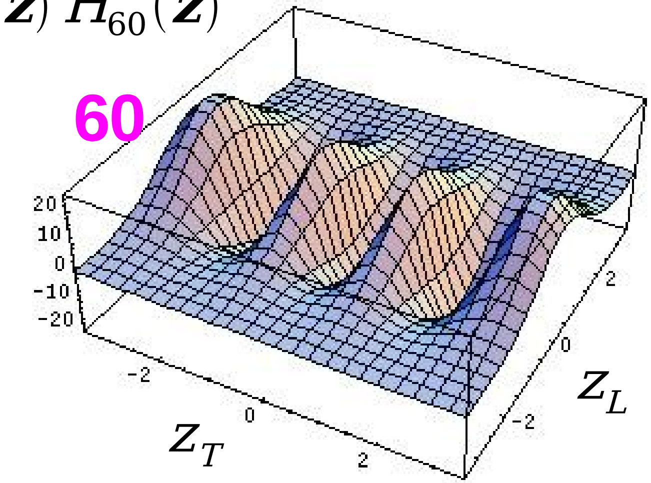
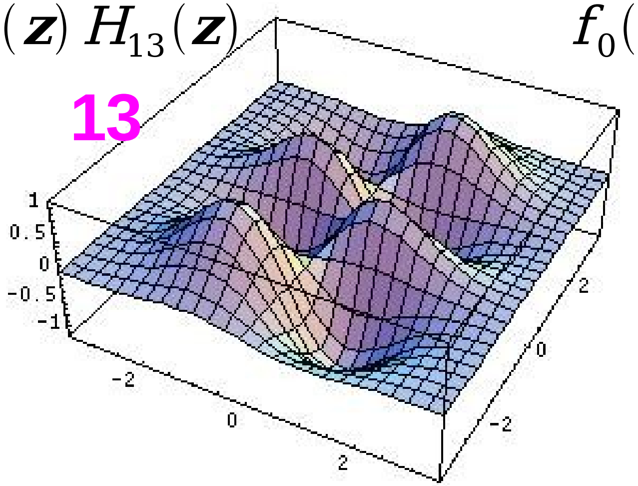
$f_0(\boldsymbol{z})\, H_{60}(\boldsymbol{z})$



60

$f_0(\boldsymbol{z})\, H_{13}(\boldsymbol{z})$



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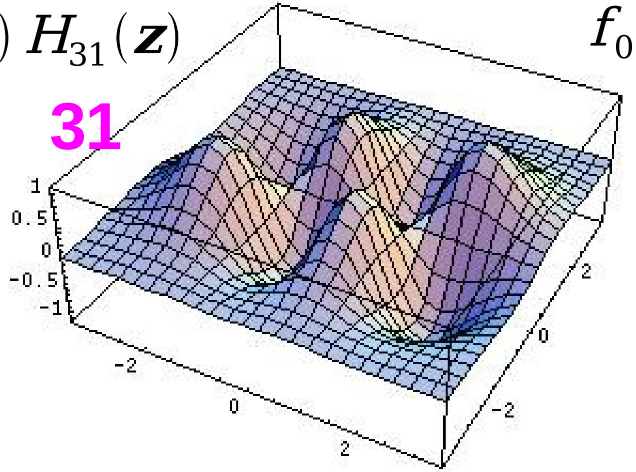
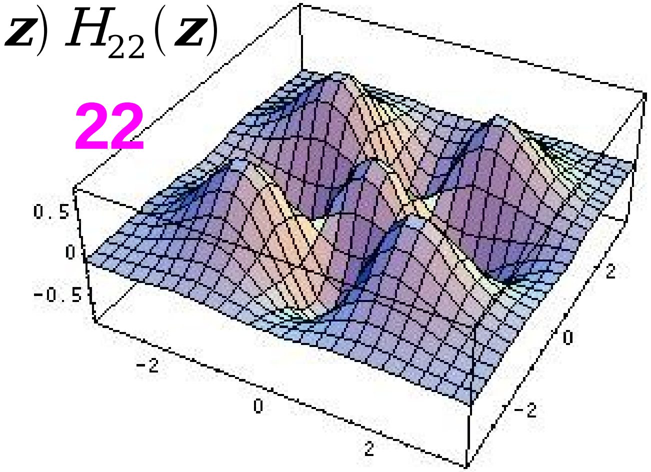
$f_0(\boldsymbol{z})\, H_{31}(\boldsymbol{z})$



31

$f_0(\boldsymbol{z})\, H_{22}(\boldsymbol{z})$



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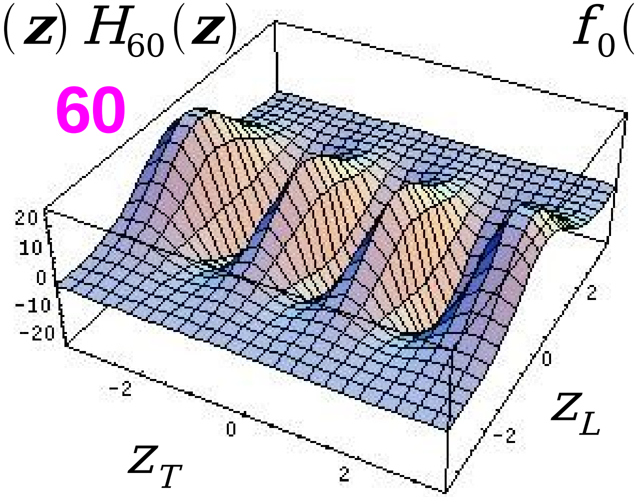
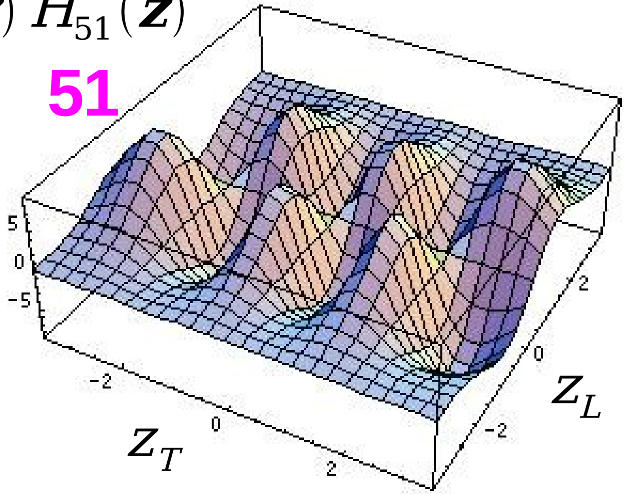
Hermite tensors: 6 th order in (T,L)

$\boldsymbol{z}=(z_T,z_L)$ $\qquad z_i=\dfrac{q_i}{\sigma_i}$ ie everything is scaled by gaussian's widths

$f_0(\boldsymbol{z})\,H_{60}(\boldsymbol{z})$

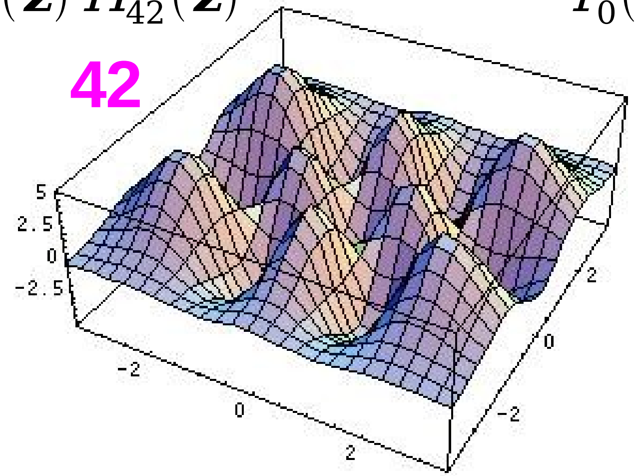


$f_0(\boldsymbol{z})\,H_{51}(\boldsymbol{z})$

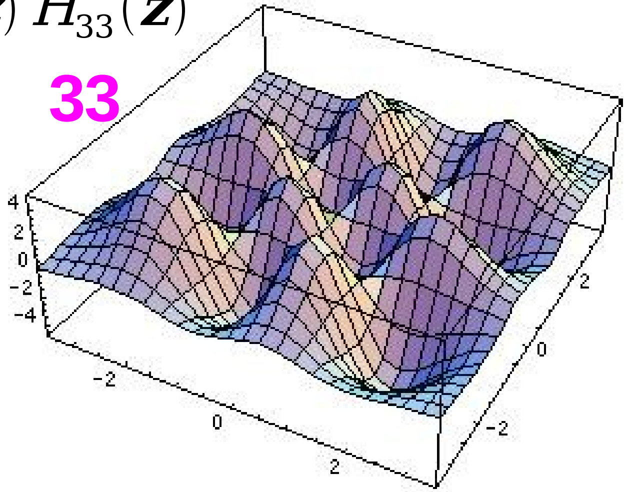


Higher-order cumulants probe the tails of the distribution

$f_0(\boldsymbol{z})\,H_{42}(\boldsymbol{z})$



$f_0(\boldsymbol{z})\,H_{33}(\boldsymbol{z})$



Symmetry of indices corresponds to geometrical symmetry

Components of Edgeworth in (T,L)

$$f(z_T,z_L)=f_0(z_T,z_L)\left(1+F_{ij}C_{ij}H_{ij}\right)$$

F_{ij} = combinatoric factor

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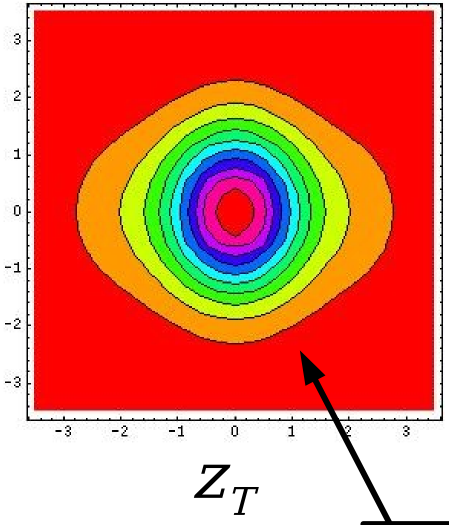

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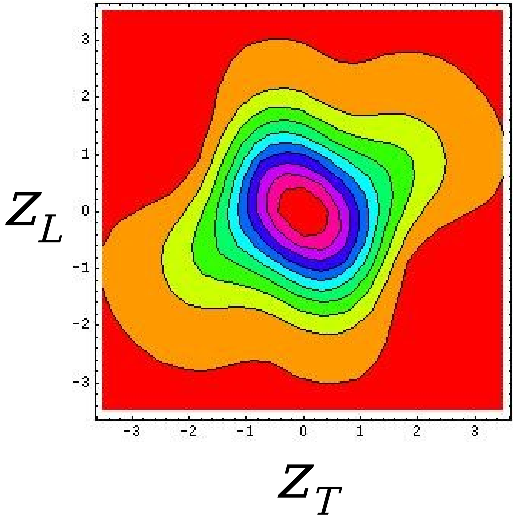

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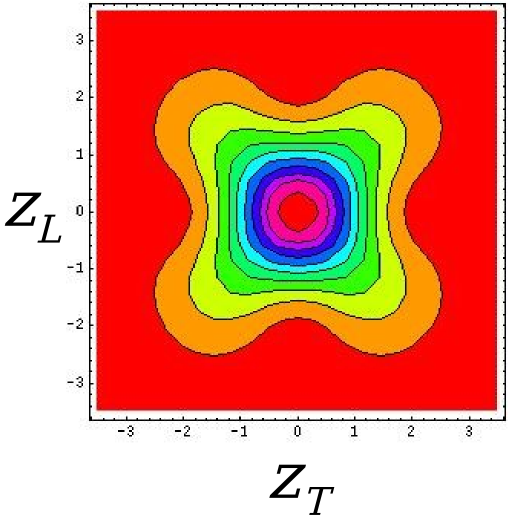

NOTE: cumulants

== probing of tails

Asymmetries appear even for ρ = 0

univariate cumulants do not influence shape much

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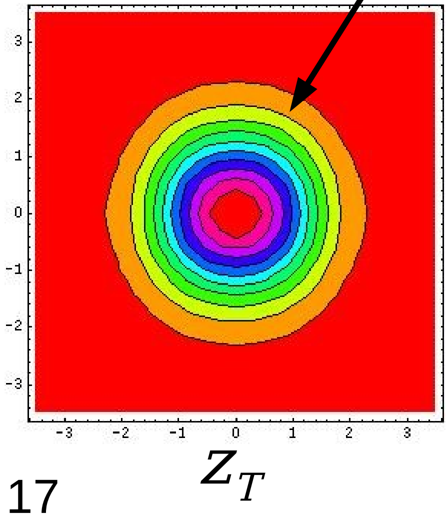

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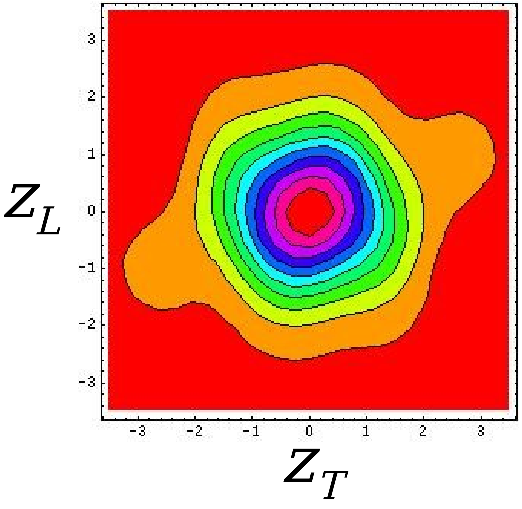

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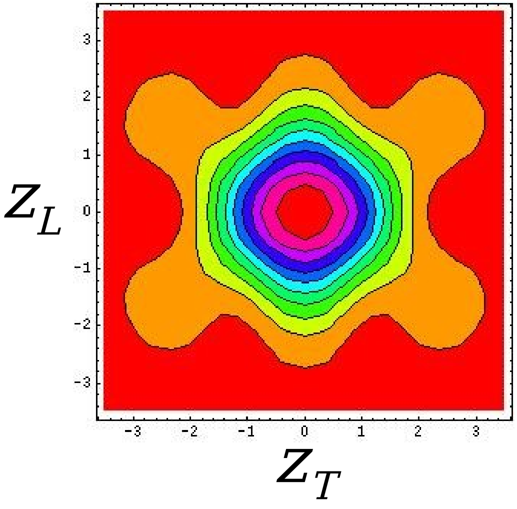

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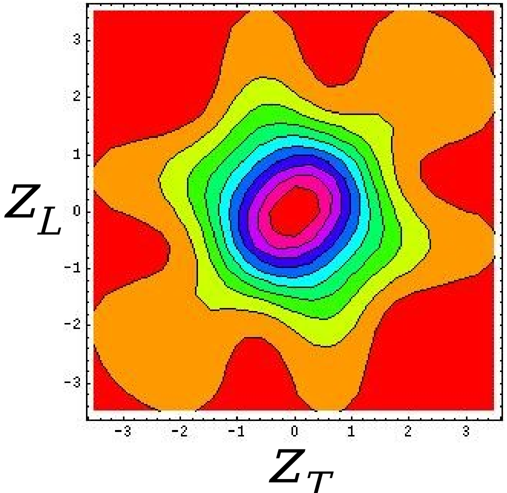
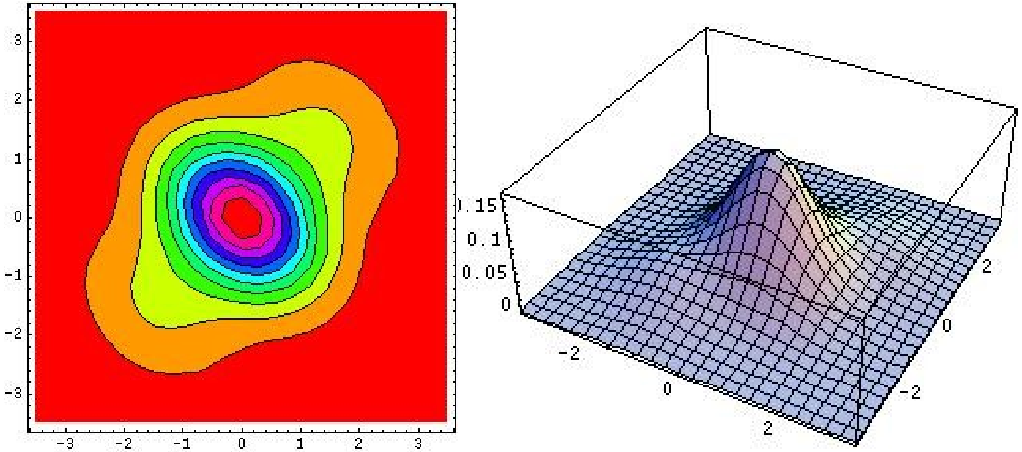
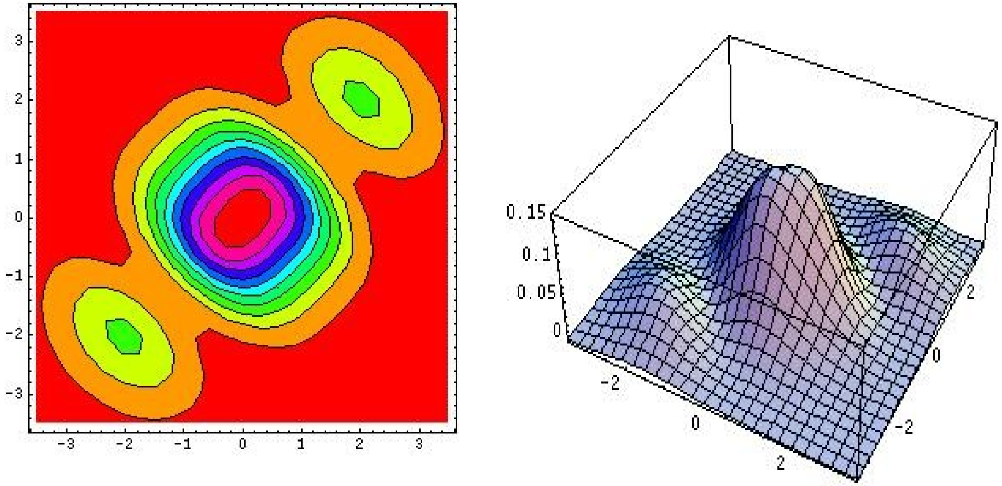


Full Edgeworth in (T,L): an arbitrary example

4th order only



4th plus 6th order



$$\frac{f(\boldsymbol{q})}{f_0(\boldsymbol{q})} = 1+\frac{1}{24}\left[\ C_{40}\,H_{40}[2]+4\,C_{31}\,H_{31}[2]+6\,C_{22}\,H_{22}\ \right]$$

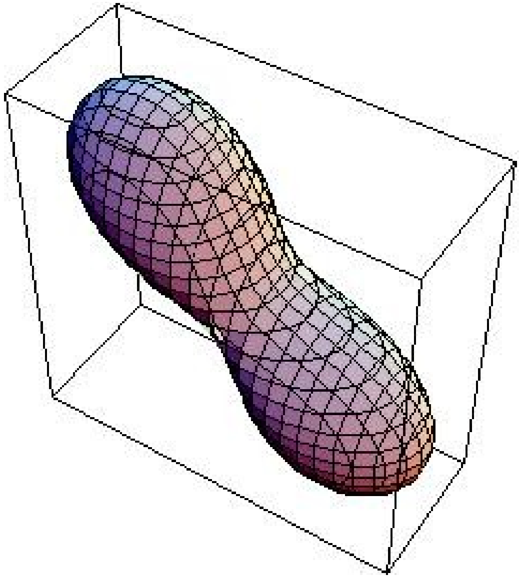
$$+\frac{1}{720}\left[\ C_{60}\,H_{60}[2]+6\,C_{51}\,H_{51}[2]+15\,C_{42}\,H_{42}[2]+20\,C_{33}\,H_{33}\ \right]$$

2d (T,L) case

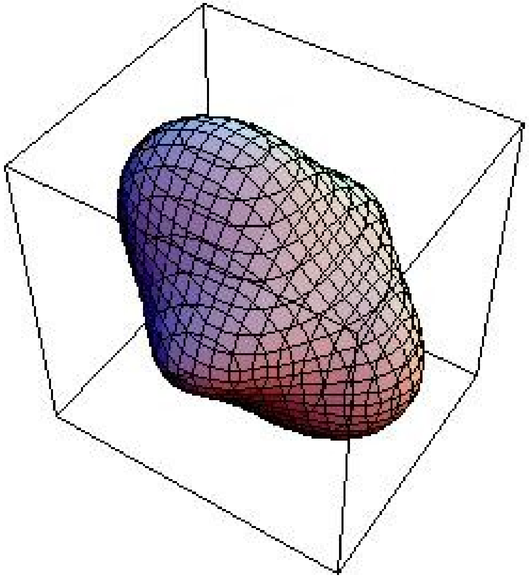
Edgeworth components in (O, S, L):
Example with one distribution, three contour values

$$f(\boldsymbol{q})=f_0(\boldsymbol{q})\left(1+12\,C_{121}\,H_{121}\right) \qquad C_{121}=0.4$$

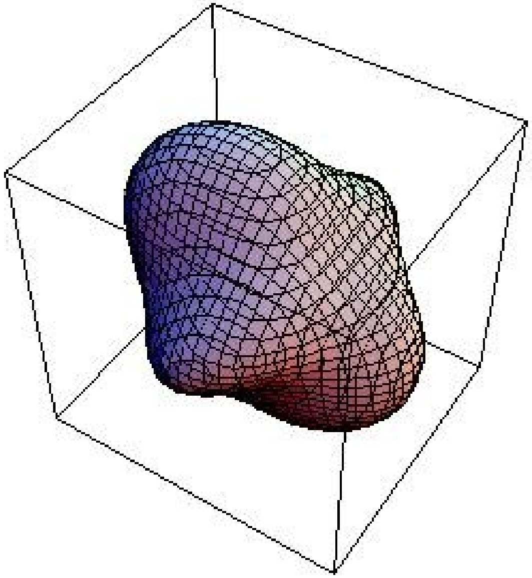
0.05 contour



0.02 contour



0.01 contour

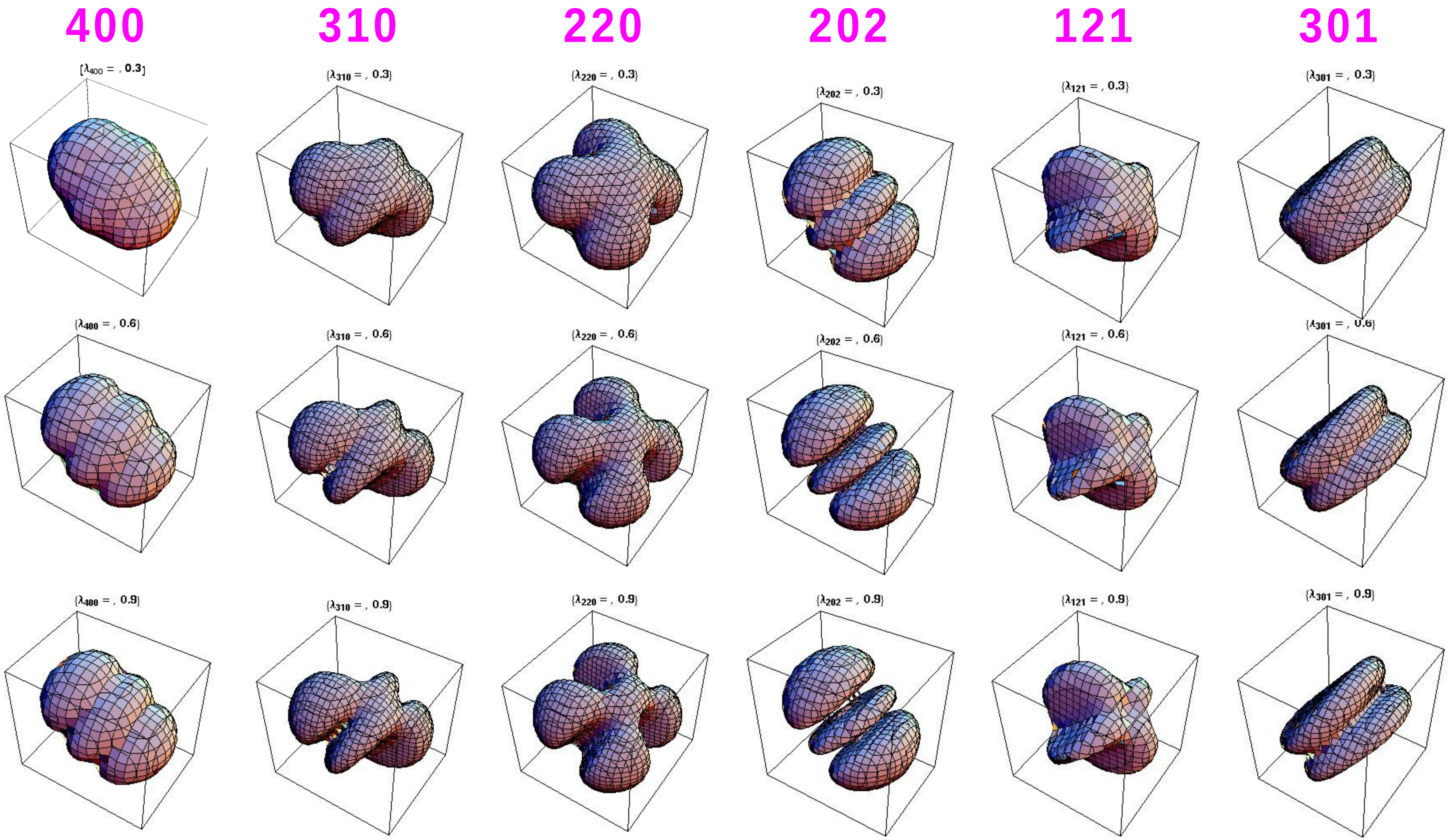


(drawings not to scale)

Evolution of Edgeworth terms with cumulant size

$$f(z_O, z_S, z_L) = f_0(z_O, z_S, z_L)\left(1 + F_{ijk} C_{ijk} H_{ijK}\right)$$

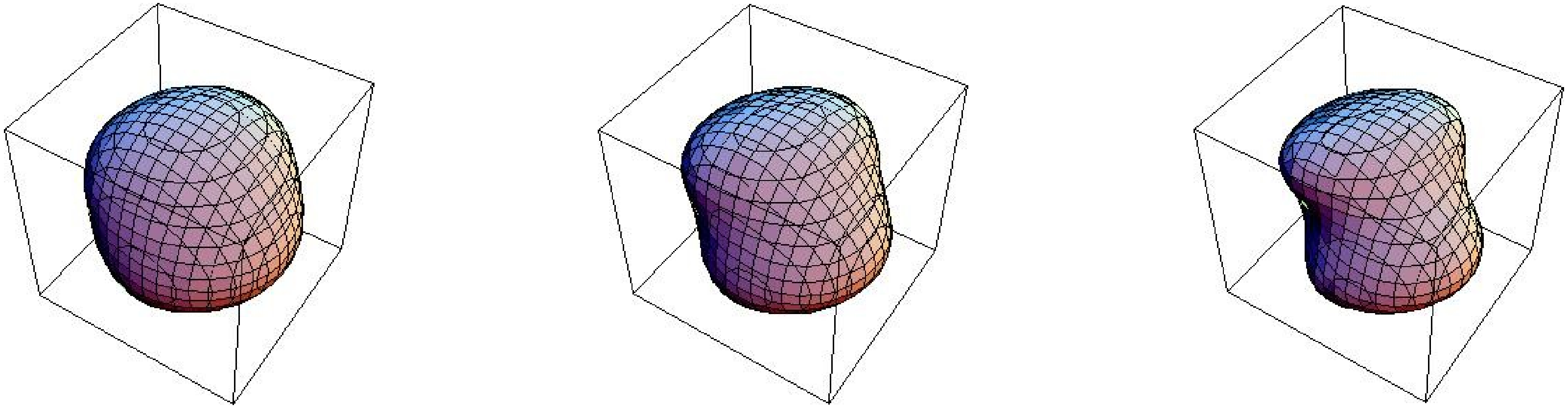
15 different cumulants



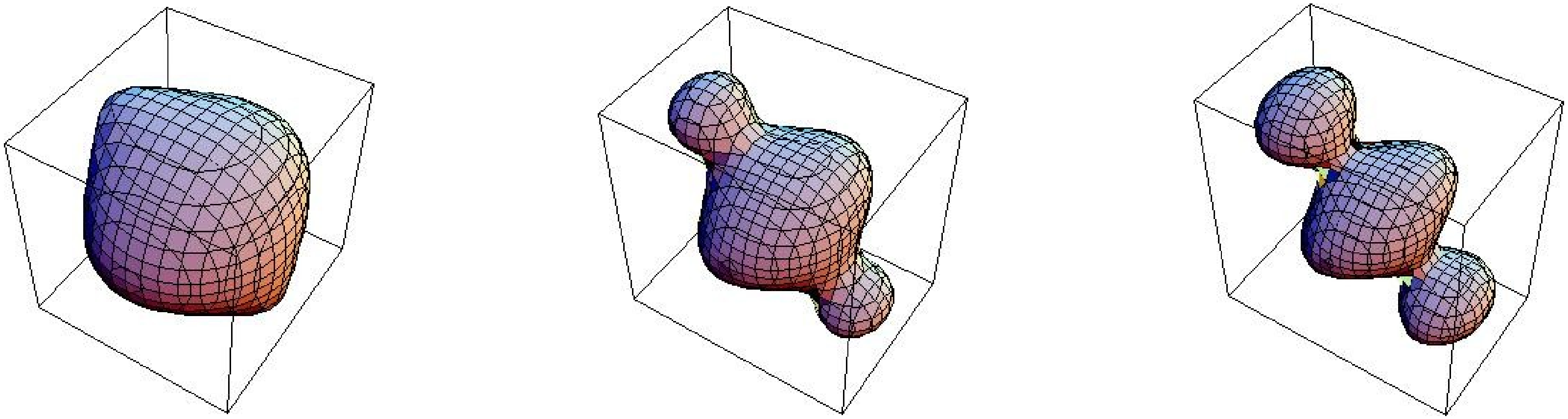
Eggers WPCF 2006

Full Edgeworth in (O,S,L): some examples

Combination of all the terms in the Edgeworth may result in rather tame final shapes . . .



. . . but then again, it may not:



Systematic characterisation of distribution shapes

- **ZERO higher-order cumulants** imply PURE GAUSSIAN, and so cumulants provide a suitable **null-measure baseline.**
- **NONZERO higher-order cumulants:**
  - In TWO dimensions, there are 5 cumulants of 4 th order, 7 of 6th order
  - In THREE dimensions, there are 15 in 4 th order, 27 in 6th order.
  - These cumulants represent the **coefficients in an n-dimensional space**
  - Since cumulants scale like <N> (in case of statistical independence), they should probably be compared in the form C_{ijk} / <N>
- **Cumulants are NUMBERS, not functions.**
  Shape comparisons are made in terms of these sets of numbers, not in terms of plots of angular coefficients vs q = |**q**|.
- Cumulants can hence be conveniently plotted as a function of pair momentum **K** or any other quantity of interest.

Possible alternative systematics

- Alternative systematics of shape: location, size, orientation etc.
- “Location” easily identified with the mean < **q** >
- “Size” is probably best couched in terms of covariance matrix eigenvalues, eg in two dimensions by $\sigma_A = \sqrt{\lambda_A} \qquad \sigma_B = \sqrt{\lambda_B}$

with

$$\lambda_A = \frac{1}{2}\left(\lambda_{TT}+\lambda_{LL}\right)+\sqrt{(\lambda_{TT}-\lambda_{LL})^2+4\lambda_{TL}^2}$$

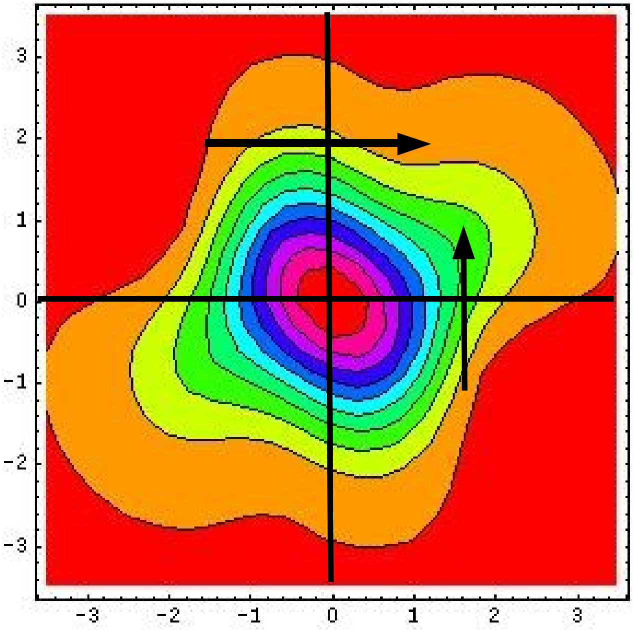
$$\lambda_B = \frac{1}{2}\left(\lambda_{TT}+\lambda_{LL}\right)-\sqrt{(\lambda_{TT}-\lambda_{LL})^2+4\lambda_{TL}^2}$$

- “Orientation” would be specified in terms of rotation angle(s) eg in two dimensions

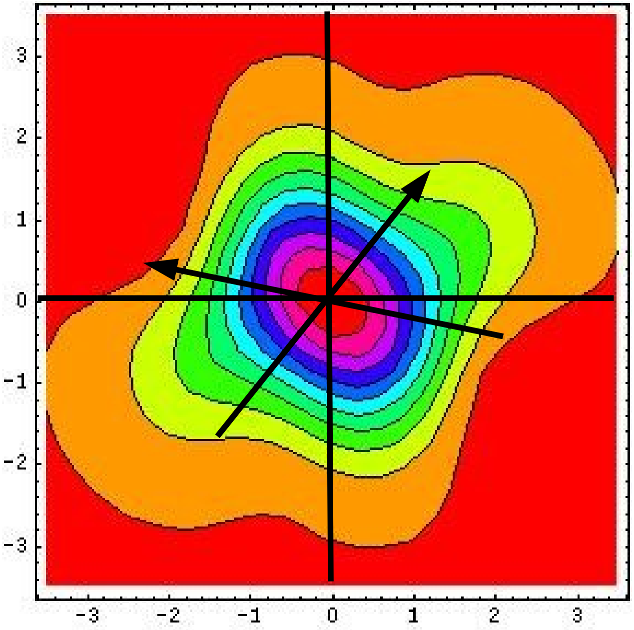
$$\theta = \frac{1}{2}\arctan\left(\frac{\lambda_{TL}}{\left|\lambda_{TT}-\lambda_{LL}\right|}\right)$$

- However, this approach decouples from the physics of the cartesian coordinate system (out, side, long).

Symmetries of correlation function



Reflection symmetry does not hold. Bins may not be combined like this.



Double up bins through the **origin**.

- Even q-parity, ie invariance of $R_2(-\boldsymbol{q}) = R_2(\boldsymbol{q})$ is a good symmetry
- In ONE dimension, (Qinv or projections), parity-symmetry is equivalent to reflection symmetry
- In TWO or THREE dimensions, reflection symmetries do not hold and may not be used for binning.
- Even for the pure gaussian case (zero higher order cumulants) a nonzero Pearson coefficient destroys reflection symmetry.
- **Cumulants of odd order must vanish** in two and three dimensions.

Direct measurement of q-cumulants

q-moments are measured directly from the normalised correlation function

$$
\begin{aligned}
[\![q_i]\!] &= \int d^3q\, f(\boldsymbol{q})\, q_i \quad \text{should be exactly zero.} \\
[\![q_i q_j]\!] &= \int d^3q\, f(\boldsymbol{q})\, q_i q_j \\
[\![q_i q_j q_k q_l]\!] &= \int d^3q\, f(\boldsymbol{q})\, q_i q_j q_k q_l
\end{aligned}
$$

Since $\kappa_i = [\![q_i]\!] \equiv 0$ the usual moment-cumulant relations simplify:

$$
\begin{aligned}
\mu_{ijkl} &= \kappa_{ijkl} + \kappa_i \kappa_{jkl}[4] + \kappa_{ij}\kappa_{kl}[3] + \kappa_i \kappa_j \kappa_{kl}[6] + \kappa_i \kappa_j \kappa_k \kappa_l \\
&= \kappa_{ijkl} + \kappa_{ij}\kappa_{kl}[3]
\end{aligned}
$$

$$
\kappa_{ijkl} = [\![q_i q_j q_k q_l]\!] - [\![q_i q_j]\!]\, [\![q_k q_l]\!]\, [3]
$$

- Measurements can be made with a correlation integral prescription (does not need binning): see below.

- (Probably) cumulants can be projected out using an orthogonal system of hermite tensors.

Other issues of experimental implementation

- The intercept parameter (chaoticity) λ is not determined by Edgeworth; it can be determined from a one (may be two-) parameter fit using the "fit function"

$$C_2(\boldsymbol{q}) = \gamma\left[1+\lambda \exp\left(-\sum_{ij} q_i q_j R_{ij}^2\right)\left(1+\frac{1}{24}\kappa_{ijkl} h_{ijkl}(\boldsymbol{q})+\frac{1}{720}\kappa_{ijklmn} h_{ijklmn}(\boldsymbol{q})\right)\right]$$

- Overall normalisation γ in $C_2(\boldsymbol{q})=\gamma\left[1+\lambda R_2(\boldsymbol{q})\right]$ must be determined as above or at z_i = several σ

- Edgeworth expansion works only for limited deviation from gaussian shapes since the expansion can become negative. Frameworks using other reference functions $f_0(\mathbf{q})$ should be investigated (will be more complicated).

- IF cumulants are measured from binned $R_2(\mathbf{q})$, then sampling errors will have to take into account bin-bin correlations.

- For very small multiplicities, attention should be paid to unbiased estimators for the cumulants (k-statistics).

Summary

- Cumulants, hermite tensors provide framework for multivariate shape analysis.
- The **null measure is the gaussian**, characterised by **6 cumulants λ_{ij}** (reducing to 4 for azimuthal symmetry). **These can be measured directly: no fits**.
- **Stepwise reconstruction** of $R_2(\mathbf{q})$: gaussian, 4th order, 6th order cumulants
- Higher-order cumulants are **not fits** but are **measured directly**
- **Off-diagonal cumulants dominate the shape**
- Shapes can be **asymmetric even for zero R_{OL} (zero ρ)** due to asymmetric κ_{ijkl}
- Hence folding of binning is not permitted except through parity
- **Lower-order quantities ($\langle \mathbf{q} \rangle$, third order) really must be zero** for higher-order shape analysis to work.
- No difficulty in principle with **extension to source function analysis**
- Framework can be extended to nonidentical particles
- However: **Edgeworth allows only limited deviation from gauss, hence this cannot be applied to highly nongaussian correlation functions.**

Multivariate moments and cumulants

$$\boxed{1}\quad \mu^{i} = \kappa^{i} \qquad\qquad \boxed{2}\quad \mu^{ij} = \kappa^{ij} + \kappa^{i}\kappa^{j}$$

$$\boxed{3}\quad \mu^{ijk} = \kappa^{ijk} + \overbrace{\kappa^{i}\kappa^{jk} + \kappa^{j}\kappa^{ik} + \kappa^{k}\kappa^{ij}}^{\text{three terms in combinatorics}} + \kappa^{i}\kappa^{j}\kappa^{k}$$

$$= \kappa^{ijk} + \underbrace{\kappa^{i}\kappa^{jk}[3]}_{\text{combinatorics notation}} + \kappa^{i}\kappa^{j}\kappa^{k}$$

$$\boxed{4}\quad \mu^{ijkl} = \kappa^{ijkl} + \kappa^{i}\kappa^{jkl}[4] + \kappa^{ij}\kappa^{kl}[3] + \kappa^{i}\kappa^{j}\kappa^{kl}[6] + \kappa^{i}\kappa^{j}\kappa^{k}\kappa^{l}$$

$$\boxed{6}\quad \mu^{ijklmn} = \kappa^{ijklmn} + \kappa^{i}\kappa^{jklmn}[6] + \kappa^{ij}\kappa^{klmn}[15] + \kappa^{i}\kappa^{j}\kappa^{klmn}[15]$$

$$+ \kappa^{ijk}\kappa^{lmn}[10] + \kappa^{i}\kappa^{jk}\kappa^{lmn}[60] + \kappa^{i}\kappa^{j}\kappa^{k}\kappa^{lmn}[20] + \kappa^{ij}\kappa^{kl}\kappa^{mn}[15]$$

$$+ \kappa^{i}\kappa^{j}\kappa^{kl}\kappa^{mn}[45] + \kappa^{i}\kappa^{j}\kappa^{k}\kappa^{l}\kappa^{mn}[15] + \kappa^{i}\kappa^{j}\kappa^{k}\kappa^{l}\kappa^{m}\kappa^{n}$$

For the special case $\kappa_{i} = 0 \qquad \kappa_{ij} = 0$

$$\boxed{3}\quad \mu^{ijk} = \kappa^{ijk}$$

$$\boxed{4}\quad \mu^{ijkl} = \kappa^{ijkl}$$

$$\boxed{6}\quad \mu^{ijklmn} = \kappa^{ijklmn} + \kappa^{ijk}\kappa^{lmn}[10]$$

Correlation-integral measurement of cumulants

a,b = event indices i,j = components of **q**

α,β = track indices

$$C = \int d^3 q \left[C_2(\boldsymbol{q}) - 1 \right]$$

$$f(\boldsymbol{q}) = \frac{\left[C_2(\boldsymbol{q}) - 1 \right]}{C} = \frac{\left\langle \sum_{\alpha \neq \beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{aa}_{\alpha\beta}) \right\rangle_a - \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{ab}_{\alpha\beta}) \right\rangle_b \right\rangle_a}{C \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{ab}_{\alpha\beta}) \right\rangle_b \right\rangle_a}$$

eventwise probability:

$$\hat{f}(\boldsymbol{q}) = \frac{\sum_{\alpha \neq \beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{aa}_{\alpha\beta}) - \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{ab}_{\alpha\beta}) \right\rangle_b}{C \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{ab}_{\alpha\beta}) \right\rangle_b \right\rangle_a}$$

$$\llbracket q_i \rrbracket = \left\langle \hat{f}(\boldsymbol{q})\, q_i \right\rangle_a = \frac{\left\langle \sum_{\alpha \neq \beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{aa}_{\alpha\beta})\, q_i \right\rangle_a - \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{ab}_{\alpha\beta})\, q_i \right\rangle_b \right\rangle_a}{C \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{ab}_{\alpha\beta}) \right\rangle_b \right\rangle_a}$$

$$\llbracket q_i q_j \rrbracket = \left\langle \hat{f}(\boldsymbol{q})\, q_i q_j \right\rangle_a = \frac{\left\langle \sum_{\alpha \neq \beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{aa}_{\alpha\beta})\, q_i q_j \right\rangle_a - \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{ab}_{\alpha\beta})\, q_i q_j \right\rangle_b \right\rangle_a}{C \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q} - \boldsymbol{q}^{ab}_{\alpha\beta}) \right\rangle_b \right\rangle_a}$$

Correlation-integral measurement of q-cumulants

- Accumulate **sibling-pair counters** on a pair-by-pair basis:

$$\rho(\boldsymbol{q}) = \left\langle \sum_{\alpha\neq\beta} \delta(\boldsymbol{q}-\boldsymbol{q}_{\alpha\beta}^{aa}) \right\rangle_a$$

$$\rho_i(\boldsymbol{q}) = \left\langle \sum_{\alpha\neq\beta} \delta(\boldsymbol{q}-\boldsymbol{q}_{\alpha\beta}^{aa})\, q_i \right\rangle_a$$

$$\rho_{ij}(\boldsymbol{q}) = \left\langle \sum_{\alpha\neq\beta} \delta(\boldsymbol{q}-\boldsymbol{q}_{\alpha\beta}^{aa})\, q_i q_j \right\rangle_a$$

- Similarly accumulate **mixed-event counters**

$$\rho^{\mathrm{ref}}(\boldsymbol{q}) = \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q}-\boldsymbol{q}_{\alpha\beta}^{ab}) \right\rangle_b \right\rangle_a$$

$$\rho_i^{\mathrm{ref}}(\boldsymbol{q}) = \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q}-\boldsymbol{q}_{\alpha\beta}^{ab})\; q_i \right\rangle_b \right\rangle_a$$

$$\rho_{ij}^{\mathrm{ref}}(\boldsymbol{q}) = \left\langle \left\langle \sum_{\alpha,\beta} \delta(\boldsymbol{q}-\boldsymbol{q}_{\alpha\beta}^{ab})\; q_i q_j \right\rangle_b \right\rangle_a$$

- **Combine counters** to form appropriate cumulants after the event loop as in previous slide. As all q's are done pair by pair, no Sheppard corrections will be necessary. Eg.

$$[\![q_i q_j]\!] = \frac{\rho_{ij}(\boldsymbol{q}) - \rho_{ij}^{\mathrm{ref}}(\boldsymbol{q})}{C\;\; \rho^{\mathrm{ref}}(\boldsymbol{q})}$$