Dirac's hole theory versus quantum field theory

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Abstract: Dirac’s hole theory and quantum field theory are usually considered equivalent to each other. The equivalence, however, does not necessarily hold, as we discuss in terms of models of a certain type. We further suggest that the equivalence may fail in more general models. This problem is closely related to the validity of the Pauli principle in intermediate states of perturbation theory.

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Résumé: On considère généralement comme équivalentes la théorie des trous de Dirac et la théorie quantique des champs. Cette équivalence ne tient pas nécessairement lorsqu'on discute de modèles de certains types. Nous suggérons que cette équivalence n’existe pas non plus pour d’autres modèles plus généraux. Ce problème est étroitement lié à la validité du principe de Pauli dans les états intermédiaires de la théorie des perturbations.

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1. Introduction

Dirac’s hole theory (HT) [1] and quantum field theory (QFT) are usually considered equivalent to each other. This equivalence, however, does not necessarily hold for models of a certain type, as noted recently [2]. The purpose of this paper is to elaborate on the mechanism of this possible inequivalence between HT and QFT with more illustrations and further suggest that such inequivalence may be prevalent beyond the models of the type that we considered earlier. We start with the Dirac equation for a particle in a given potential in single-particle quantum mechanics. We examine how the energy eigenvalues of the Dirac equation vary when an external perturbation is applied. Then we consider a HT model that is based on the same Dirac equation. We also examine its QFT version. We focus on the vacuum state and its energy shift caused by an external perturbation.

The possible inequivalence between HT and QFT is closely related to the validity of Feynman’s prescription to disregard the Pauli principle (PP) in intermediate states of perturbation theory [3]. This
prescription is based on Feynman’s observation that effects of all virtual processes that violate PP cancel out (at least formally). In HT, however, the PP-violating terms do not necessarily cancel, as explicitly illustrated in ref. 4. In such a case, the result of perturbation calculation differs depending on whether or not PP is enforced in intermediate states. Then the question arises: Should we enforce PP or not?

Cavalcanti [5] found the exact solution of the HT model of ref. 4 in its special case with the particle mass \( m = 0 \). He pointed out that the exact HT solution is consistent with the perturbation calculation in which PP is disregarded throughout but not with the one in which PP is enforced. This was puzzling in view of the usual belief that PP operates in intermediate states. Cavalcanti’s comment was responded to in ref. 2, but the above puzzling aspect and the difference between HT and QFT were not fully addressed in ref. 2. In the present paper we emphasize that for models of a certain type, including that of ref. 4, HT and QFT are not necessarily equivalent, and we explore various aspects of the difference. Quantum field theory is free from the puzzling aspect that was found in refs. 4 and 5. Unlike in HT, the cancellation of the PP-violating terms in perturbation theory is complete in QFT. Finally we extend the analysis to models that involve particle–particle interactions and argue that the inequivalence between HT and QFT may be quite ubiquitous.

In Sect. 2 we specify the type of models that we consider. We discuss some difficulties of the HT version of the model. In Sect. 3 we examine the QFT version of the model and clarify the difference between HT and QFT. We examine examples in Sect. 4. The results are summarized and extension to models involving particle–particle interactions is discussed in Sect. 5.

2. The hole theory

We start with single-particle relativistic quantum mechanics with the Hamiltonian

\[
H = H_0 + V
\]  

where \( H_0 \) is the Dirac Hamiltonian for a particle in a binding potential and \( V \) is an external perturbation. The Dirac equations for the unperturbed and perturbed systems, respectively, are

\[
H_0 \phi_n = \epsilon_n \phi_n, \quad H \psi_n = \eta_n \psi_n
\]

Suffix \( n (= \pm 1, \pm 2, \ldots) \) specifies eigenstates. The energy eigenvalues are labeled such that \( 0 < \epsilon_1 < \epsilon_2 < \ldots \) and \( 0 > \epsilon_{-1} > \epsilon_{-2} > \ldots \), and similarly for \( \eta_n \). We assume that the energy levels are all discrete and non-degenerate and that there is a one-to-one correspondence between unperturbed and perturbed eigenstates. In other words, \( \eta_n \rightarrow \epsilon_n \) when \( V \rightarrow 0 \). Let us also assume that \( \eta_n \) and \( \epsilon_n \) have the same sign. It is not difficult to relax these restrictions.

We now turn to HT and consider the vacuum in which all negative-energy states are occupied. The unperturbed and perturbed energies of the vacuum are, respectively, given by \( E_0 = \sum_j \epsilon_{-j} \) and \( E = \sum_j \eta_{-j} \). We are interested in the energy shift due to the perturbation

\[
\Delta E = \sum_j (\eta_{-j} - \epsilon_{-j})
\]

Suppose that we start with known \( \epsilon_n \) and attempt to reach \( \eta_n \) by perturbation theory. We treat the system as a one-body system or, more precisely, as an ensemble of independent one-body systems. We follow the standard prescription of perturbation theory including all intermediate states. We do not exclude the intermediate states that are already occupied by other particles. The first-order energy shift is given by

\[
\Delta E^{(1)} = \sum_j V_{-j,-j} = \int V(r) \rho_{\text{vac}}(r) \, dr
\]
where $V_{-j,-j} = \langle \phi_{-j}|V|\phi_{-j} \rangle$ and $\rho_{\text{vac}}(r) = \sum_j |\phi_{-j}(r)|^2$ is the particle density of the unperturbed vacuum. This $\Delta E^{(1)}$ as such generally diverges because $\rho_{\text{vac}}(r)$ is actually infinite. No matter how weak it is, $V$ may cause an infinite energy shift. This difficulty is avoided by assuming that the density in the perturbed vacuum itself is not an observable quantity and that only the difference between the density of the perturbed vacuum and its unperturbed counterpart is observable. Then the first-order energy shift disappears.

We are more interested in the second-order energy shift

$$\Delta E^{(2)} = \sum_j \Delta \epsilon^{(2)}_{-j} = \sum_j \frac{|V_{i,-j}|^2}{\epsilon_{-j} - \epsilon_i} + X$$

$$X = \sum_j \sum_{k \neq j} \frac{|V_{-k,-j}|^2}{\epsilon_{-j} - \epsilon_{-k}}$$

(6)

where $V_{i,-j} = \langle \phi_i|V|\phi_{-j} \rangle$. Term $X$ is due to the transitions between negative-energy states like the one from $-j$ to $-k$. In $\Delta E^{(2)}$, PP has not been considered. If we enforce PP, the transitions between negative energy states are not allowed and we obtain

$$\Delta E^{(2)}_{\text{PP}} = \sum_{i,j} \frac{|V_{i,-j}|^2}{\epsilon_{-j} - \epsilon_i}$$

(7)

where suffix PP means “with PP enforced”. The $X$ of (6) is an infinite alternating series. This formally vanishes because the numerator is symmetric with respect to $j \rightarrow k$ while the denominator is antisymmetric. As was illustrated in ref. 4, however, $X$ may not vanish depending on how the calculation is done. We discuss more illustrations in Sect. 4.

The question that we face is: If $X$ does not vanish, should we include it or not? According to the exactly solvable HT example in ref. 5, which we will discuss in detail in Sect. 4, $X$ should be included. If one enforces PP in intermediate states, one obtains a wrong energy shift in HT. This is puzzling. According to the spin-statistics theorem (which was proved in QFT), the wave function of a fermion system has to be antisymmetric with respect to interchanges of the particles. This implies PP. The perturbative interaction $V$ is totally symmetric with respect to the interchange of particles. It acts in the same way on all particles. Then the intermediate state that is connected to the antisymmetric initial state through $V$ has to be antisymmetric. Hence PP is expected to hold in the intermediate states. This is also related to the unitarity of the $S$ matrix [6, 7]. Recall also that Dirac introduced HT such that the vacuum state is stable. This stability relies on PP.

3. Quantum field theory

In QFT the Hamiltonian for the unperturbed system can be expressed as

$$\mathcal{H}_0 = \sum_i \epsilon_i a_i^\dagger a_i + \sum_j \bar{\epsilon}_j b_j^\dagger b_j, \quad \bar{\epsilon}_j = -\epsilon_{-j} > 0$$

(8)

The notation is hopefully self-explanatory. The $a_i$ ($a_i^\dagger$) is the creation (annihilation) operator for the particle in the unperturbed state $i$. The $a_i^\dagger$ creates a particle with energy $\epsilon_i$ with wave function $\phi_i(x)$. These operators satisfy the usual anticommutation relations. The $b_j$ ($b_j^\dagger$) is for the antiparticle. No negative-energy particles appear in QFT. Let us emphasize that the Hamiltonian is defined in terms of normal products of the creation and annihilation operators. The unperturbed vacuum contains no particles nor antiparticles. It is the eigenstate of $\mathcal{H}_0$ with zero eigenvalue.
The interaction Hamiltonian is of the form of

$$V = \sum_{i,i'} V_{i,i'} a_i^{+} a_{i'} + \sum_{i,j} V_{i,-j} a_i^{+} b_j^{+} + \sum_{i,j} V_{-j,i} b_j a_i - \sum_{j,j'} V_{-j,j'} b_j^{+} b_{j'}$$  \hspace{1cm} (9)$$

where, for example, $a_i^{+} b_j^{+}$ creates a pair of particles and antiparticles. The negative sign of the last term arises when $b b^{+}$ is rearranged into the normal product $b^{+} b$. The number of the particles is conserved with the understanding that an antiparticle has particle number $-1$. Because $V$ consists of normal products, its expectation value in the unperturbed vacuum is zero. Hence we obtain

$$\Delta E^{(1)} = 0$$  \hspace{1cm} (10)$$

We use notation $E$ for the energy of the system in QFT. Equation (10) is in a sharp contrast to (4). Quantum field theory is free from the difficulty of the infinite first-order energy shift of HT that we pointed out below (4). If we did not use the normal products for $V$, we would obtain (4) for $\Delta E^{(1)}$.

Next let us examine the energy shift of the second order. In QFT the vacuum does not contain any particles or antiparticles. The transitions between negative-energy states of HT has no place in QFT. We obtain

$$\Delta E^{(2)} = - \sum_{i,j} \frac{|V_{i,-j}|^2}{\epsilon_i + \bar{\epsilon}_j}$$  \hspace{1cm} (11)$$

which agrees with $\Delta E_{pp}^{(2)}$ of HT. Recall that $\Delta E_{pp}^{(2)}$ and $\Delta E^{(2)}$ may or may not agree with each other in HT.

If the exact solutions of the perturbed Dirac equation are known, they can be used to define the creation and annihilation operators $c$, $c^{+}$, $d$, and $d^{+}$ for the perturbed system. The Hamiltonian for the perturbed system then becomes

$$\mathcal{H} = \sum_{i} \eta_i c_i^{+} c_i + \sum_{j} \bar{\eta}_j d_j^{+} d_j, \hspace{1cm} \bar{\eta}_j = -\eta_{-j} > 0$$  \hspace{1cm} (12)$$

The perturbed vacuum is the eigenstate of $\mathcal{H}$ with zero eigenvalue. The perturbed vacuum contains no particles or antiparticles that are defined in terms of $\{c, c^{+}, d, d^{+}\}$.

In the absence of perturbation, the vacuum energy is zero. In the presence of perturbation, if we use $\mathcal{H}$ given above, the vacuum energy is again zero. This may give the false impression that the perturbation causes no energy shift. Recall that the Hamiltonian is defined as a normal product. The normal product depends on how the creation and annihilation operators are defined. This dependence gives rise to the energy shift. This can be seen as follows. We have two sets of basis functions, $\{\phi_n\}$ and $\{\psi_n\}$, each of which forms a complete orthonormal system. The $\{a, a^{+}, b, b^{+}\}$ and $\{c, c^{+}, d, d^{+}\}$ are related by the Bogoliubov transformation

$$c_i = \sum_{i'} \langle \psi_i | \phi_{i'} \rangle a_{i'} + \sum_{j} \langle \psi_i | \phi_{-j} \rangle b_j^{+}$$  \hspace{1cm} (13)$$

$$d_j^{+} = \sum_{i} \langle \psi_{-j} | \phi_i \rangle a_i + \sum_{j'} \langle \psi_{-j} | \phi_{-j'} \rangle b_{j'}^{+}$$  \hspace{1cm} (14)$$

and their hermitian adjoints. If we rewrite $\mathcal{H}_0 + V$ in terms of $\{c, c^{+}, d, d^{+}\}$, we expect to obtain

$$\mathcal{H}_0 + V = \mathcal{H} + \Delta E$$  \hspace{1cm} (15)$$
The $\Delta E$ is the expectation value of $\mathcal{H}_0 + \mathcal{V}$ in the perturbed vacuum. The $\Delta E$ is also the negative of the expectation value of $\mathcal{H}$ in the unperturbed vacuum. If $\mathcal{V} \to 0$, then $\mathcal{H} \to \mathcal{H}_0$ and $\Delta E \to 0$. Therefore, $\Delta E$ is the energy shift of the vacuum due to the perturbation $\mathcal{V}$.

Equation (15) has various interesting implications. Let us first work out $\Delta E$ explicitly. It is somewhat simpler to start with $\mathcal{H}$ and rewrite it in terms of $\{a, a^\dagger, b, b^\dagger\}$. The combination $c_i^* c_i$ goes like

$$c_i^* c_i = \left( \sum_{i'} \langle \psi_i | \phi_{i'} \rangle^* a_{i'}^\dagger + \sum_j \langle \psi_i | \phi_{-j} \rangle^* b_j \right) \left( \sum_{i''} \langle \psi_i | \phi_{i''} \rangle a_{i''} + \sum_{j'} \langle \psi_i | \phi_{-j'} \rangle^* b_{j'} \right)$$

$$= \sum_{i',i''} \langle \psi_i | \phi_{i'} \rangle^* \langle \psi_i | \phi_{i''} \rangle a_{i'}^\dagger a_{i''} - \sum_{j,j'} \langle \psi_i | \phi_{-j} \rangle^* \langle \psi_i | \phi_{-j'} \rangle b_j b_{j'} + \sum_j \langle \psi_i | \phi_{-j} \rangle^* \langle \psi_i | \phi_{-j} \rangle \right|^2 + \ldots$$

(16)

where $\ldots$ indicates terms of the form of $a^\dagger b^\dagger$ and $ba$. The combination $d_i^* d_i$ goes similarly. If we substitute the above into $\mathcal{H}$ and collect the terms of the form of, e.g., $a^\dagger a$, we obtain

$$\sum_{i',i''} \left( \sum_i \eta_i \langle \psi_i | \phi_{i'} \rangle^* \langle \psi_i | \phi_{i''} \rangle - \sum_j \bar{\eta}_{-j} \langle \psi_{-j} | \phi_{i'} \rangle \langle \psi_{-j} | \phi_{i''} \rangle^* \right) a_{i'}^\dagger a_{i''}$$

$$= \sum_{i,i'} (\langle H_0 + \mathcal{V} | \phi_{i'} \rangle a_{i'}^\dagger a_i = \sum_i \epsilon_i a_i^\dagger a_i + \sum_{i,i'} V_{i,i'} a_i^\dagger a_{i'}$$

(17)

We have used

$$H = H_0 + \mathcal{V} = \sum_i \eta_i \langle \psi_i | \psi_i \rangle - \sum_j \bar{\eta}_{-j} \langle \psi_{-j} | \psi_{-j} \rangle$$

(18)

The normal products of $a$, $a^\dagger$, $b$ and $b^\dagger$ altogether become $\mathcal{H}_0 + \mathcal{V}$ and we obtain

$$\Delta E = - \sum_{i,j} \left( \eta_i |\langle \psi_i | \phi_{-j} \rangle|^2 + \bar{\eta}_{-j} |\langle \psi_{-j} | \phi_i \rangle|^2 \right)$$

(19)

We believe (19) is new. The $\Delta E$ can vanish only if $\mathcal{V} = 0$. Otherwise it is negative. Let us emphasize that the sign of $\Delta E$ is not related to the sign of the single-particle energy shift $\eta - \epsilon$. Even if $\eta - \epsilon$ is positive for all levels, $\Delta E$ is negative. The vacuum is the ground state. It is well known that the ground-state energy can only go down in second-order perturbation theory. It is remarkable that the exact vacuum energy can only go down.

Let us examine the leading term of $\Delta E$ in the perturbation expansion. It is obvious that $\Delta E^{(1)} = 0$. For the matrix elements involved, we obtain in first order

$$\langle \psi_i | \phi_{-j} \rangle = \frac{V_{i,-j}}{\epsilon_i - \epsilon_{-j}}, \quad \langle \psi_{-j} | \phi_i \rangle = \frac{V_{-j,i}}{\epsilon_{-j} - \epsilon_i}$$

(20)

As far as the leading term of $\Delta E$ is concerned, we can put $\eta_i = \epsilon_i$ and $\bar{\eta}_{-j} = -\epsilon_{-j}$. Then (11) follows. One can also work out higher order energy shifts $\Delta E^{(3)}$, etc., successively.

Let us discuss a question that may arise here. Assume that $V(x)$ of (1) is attractive and it changes the energy as $\epsilon_n \to \eta_n < \epsilon_n$. Suppose we start with $\mathcal{H}$ and go backward, that is, add $-V(x)$ to $\mathcal{H}$ so that $\mathcal{H} \to \mathcal{H}_0$. Then obviously the energy changes as $\eta_n \to \epsilon_n > \eta_n$. This is single-particle quantum mechanics. In QFT the situation of the vacuum is different. When the Hamiltonian changes from $\mathcal{H}_0$ to $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$, the vacuum energy goes down irrespective of the sign of $\mathcal{V}$. Suppose one starts with $\mathcal{H}$ and the vacuum with respect to $\{c, c^\dagger, d, d^\dagger\}$. Add an interaction to $\mathcal{H}$ such that it counteracts $\mathcal{V}$. This

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interaction is in the form of normal products of $\{c, c^\dagger, d, d^\dagger\}$. The vacuum energy again decreases. This may sound strange but there is no contradiction.

The problem regarding PP in intermediate states does not arise in the QFT calculation for the vacuum energy. There is nothing that blocks the particle and antiparticle pair creation in the second-order intermediate states. Because the transitions between negative-energy states do not appear in QFT, the perturbation calculation is free from the ambiguity of the kind that was pointed out in ref. 4. In QFT (but not in HT) all PP-violating effects cancel, and hence one can disregard PP throughout, as Feynman correctly advocated.

Let us point out a feature of (15) that is interesting in relation to Feynman’s prescription. Consider a system that consists of a number of particles and antiparticles together with their vacuum background. The energy shift of this system is simply given by

$$
\sum_i (\eta_i - \epsilon_i) + \sum_j (\tilde{\eta}_j - \tilde{\epsilon}_j) + \Delta\mathcal{E}
$$

where the summations are for the particles and antiparticles. The $\Delta\mathcal{E}$ is the same as that of equation (19). In calculating $\eta_i - \epsilon_i$, for example, one does not have to think about the vacuum background. On the other hand $\Delta\mathcal{E}$ is the energy shift of the vacuum in the absence of the additional particles and antiparticles. The first-order energy shift can appear through $\eta_i - \epsilon_i$. As an illustration, consider the QFT version of the one-dimensional bag model of ref. 4 again. Assume that one particle is bound in state 1 and the external potential is $V(x) = \lambda x$. Then the second-order energy shift is essentially the polarizability of the system. This is determined by $\eta_1 - \epsilon_1 + \Delta\mathcal{E}$. The particle in state 1 and the vacuum background together should be interpreted as one physical system.

There is another interesting feature of (15). Compare two systems that differ in terms of the levels occupied by particles and antiparticles. The difference between the energies of the two systems is determined simply by the single-particle energies, $\eta$’s and $\tilde{\eta}$’s. The vacuum energy $\Delta\mathcal{E}$ is common to the two systems.

### 4. Examples

We discuss three examples, A, B, and C. Examples A and B are based on the one-dimensional bag model that was considered in refs. 2, 4, and 5. We review the model and extend it. Example C is new.

#### 4.1. Example A: One-dimensional bag model

The model is defined by

$$
H_0 = \alpha p + \beta [m + S(x)]
$$

$$
H = H_0 + V(x)
$$

where $m$ is the mass of the particle confined in the bag, $p = -i\hbar d/dx$, and $\alpha = \sigma_y$ and $\beta = \sigma_z$ are $2 \times 2$ Dirac matrices. The $S(x)$ is a Lorentz-scalar potential such that $S(x) = 0$ for $|x| < a$ and $S(x) = \infty$ for $|x| > a$.

We focus on the vacuum state. In this example we assume $m = 0$. First assume, as was done in ref. 4, that

$$
V(x) = \lambda x
$$

where $\lambda$ is a constant. It then follows that $V_{-j,-j} = 0$ and $\Delta E^{(1)} = 0$. The divergence difficulty of $\Delta E^{(1)}$ that we pointed out below (4) does not arise. For the second-order energy shift the HT calculation of ref. 4 was done in two ways, methods I and II. In method I, PP is enforced whenever it is applicable.
In method II, PP is disregarded throughout. Method I leads to $\Delta E^{(2)}_{pp}$ and method II to $\Delta E^{(2)}$. The explicit calculations of ref. 4 led to $\Delta E^{(2)}_{pp} < 0$ and $\Delta E^{(2)} = 0$.

Cavalcanti [5] showed that, if $m = 0$, the perturbed Dirac equation $H \psi_n = \eta_n \psi_n$ for the model of ref. 4 can be solved analytically. This enabled him to obtain the exact energy shift of the HT model. He showed that $\eta = \epsilon$. The eigenvalue $\eta$ is independent of the perturbation and so is the exact energy of the HT vacuum, that is, $\Delta E = 0$. This is consistent with $\Delta E^{(2)} = 0$ of method II but not with $\Delta E^{(2)}_{pp} < 0$ of method I.

Cavalcanti solved the Dirac equation by rewriting it as

$$\frac{dw_\pm(x)}{dx} + i(\lambda x - \eta)w_\pm(x) = 0, \quad w_\pm(x) = u(x) \pm iv(x)$$

where $u(x)$ and $v(x)$ are the upper and lower components of $\psi(x)$, respectively. His solution is of the form of $w_\pm(x) = C_\pm e^{i[(\lambda/2)x^2 - \eta x]}$ where $C_\pm$ are constants. This can be generalized to the case with an arbitrary potential $V(x)$ (but still with $m = 0$) by replacing Cavalcanti’s $w_\pm(x)$ with

$$w_\pm(x) = C_\pm e^{i[f(x) - \eta x]}, \quad \frac{df(x)}{dx} = V(x)$$

The boundary condition for the wave function is $w_\pm(\pm a) = \mp iw_\mp(\pm a)$, which leads to

$$\eta - \epsilon = \frac{1}{2a} \int_{-a}^{a} V(x)dx$$

where $\epsilon = (2n + 1)(\pi/4a)$ with $n = 0, \pm 1, \pm 2, \ldots$ is the energy eigenvalue of $H_0$. It is remarkable that the exact energy shift $\eta - \epsilon$ of the single-particle state is first order. The second-order and all higher order effects are zero.

If $V(x)$ is an odd function of $x$, then $\eta = \epsilon$. This leads to $\Delta E = 0$, that is, the exact energy of the HT vacuum is independent of $V(x)$. If $V(x)$ contains an even function part, all eigenvalues are shifted exactly by the same amount. The vacuum of HT contains an infinite number of negative-energy particles. If the energy of every particle is shifted by the same amount, the total energy of the vacuum obtains an infinite energy shift. This illustrates what we pointed out below (4).

In the QFT counterpart of the bag model of ref. 4, the second-order energy shift of the vacuum is given by $\Delta \mathcal{E}$ of (19). Its first-order part is absent. Its second-order part is negative definite. It agrees with HT with method I (with PP enforced). It disagrees with HT with method II (without PP). The HT with method II is the exact HT. The exact QFT energy shift can be explicitly calculated by using the known exact solutions of the Dirac equation of the model.

4.2. Example B: The same model as that of Example A except that $m > 0$

When $m > 0$, it is difficult to solve the perturbed Dirac equation analytically. The exact energy shift in second order, however, can be obtained by means of the Dalgarno–Lewis method [8]. This was done when $V(x) = \lambda x$ in ref. 4. The system that was examined in ref. 4 is such that there is one particle in the lowest positive energy state together with its vacuum background. Figure 3 of ref. 4 shows the second-order energy shifts calculated in two ways, $W$ by method I (with PP enforced) and $W'$ by method II. Method I agrees with QFT whereas method II agrees with exact HT. The $W$ and $W'$ are clearly different. The difference is essentially due to the difference of the vacuum energies. The difference becomes smaller as $m$ increases.

Does the difference between HT and QFT persist when we replace $V(x) = \lambda x$ with an arbitrary potential $V(x)$? Our answer to this question is affirmative. The Dalgarno–Lewis equation is not easy to solve for an arbitrary $V(x)$. However, an arbitrary $V(x)$ can be handled by the logarithmic perturbation expansion method, which is actually equivalent to the Dalgarno–Lewis method; see Sect. III of ref. 9.
The energy shift obtained in this way is a smooth function of \( m \). We know that HT and QFT disagree when \( m = 0 \). This discrepancy persists when \( m \) is increased. Therefore HT and QFT disagree for the one-dimensional bag model with any value of \( m \) and any potential \( V(x) \). The radius of the bag can be chosen arbitrarily large such that the energy spectrum becomes practically continuous.

4.3. Example C

Consider the one-dimensional model defined by

\[
H_0 = \alpha p + \beta [m + S(x)]
\]

\[
H = H_0 + \beta \Delta S(x)
\]

Note that we have potentials of the Lorentz-scalar type only. The Lorentz-transformation character of potential \( V \) of (1) can be anything. The \( S(x) \) is a confining potential such that \( S(x) \to \infty \) as \(|x| \to \infty \) but it is not the same as the \( S(x) \) of Example A. The energy eigenvalues of \( H_0 \) and \( H \) are both discrete. There is a symmetry between the positive- and negative-energy levels, i.e., \( \epsilon_n = -\epsilon_{-n} \) and \( \eta_n = -\eta_{-n} \).

Assume that \( \epsilon_n, n = 1, 2, 3, \ldots, N \), are all specified. Then one can construct, by means of the “inverse scattering technique”, a potential \( S(x) \) such that \( H_0 \) has these \( N \) eigenvalues \([10, 11]\). The \( S(x) \) so obtained actually contains \( N \) parameters that one can choose arbitrarily, that is, there is an \( N \)-parameter family of such potentials. See Example D, the Kay–Moses potentials, of ref. 10. The parameters involved are \( \kappa_n \) and \( A_n \) with \( n = 1, 2, 3, \ldots, N \). The \( \kappa_n \) is related to \( \eta_n \), whereas \( A_n \) can be chosen arbitrarily. For this arbitrariness, see also ref. 12. One can let \( N \to \infty \). Assume that this has been done. Let one of the potentials of the family be \( S(x) \) and another one be \( S(x) + \Delta S(x) \). This means that the perturbation \( \Delta S(x) \) causes no energy shift in the single-particle spectrum and hence no vacuum energy shift in HT. In QFT, however, \( \Delta S(x) \) leads to a negative-energy shift of the vacuum. This is because, although they are equivalent with respect to the energy spectrum, \( S(x) \) and \( S(x) + \Delta S(x) \) are different functions of \( x \). It is impossible that the matrix elements of \( \Delta S(x) \) that appear in (19) and (20) all vanish. This model suggests that the inequivalence between HT and QFT persists when the sharp boundary of the bag model of Examples A and B is replaced by a smooth confining potential.

Instead of the Lorentz-scalar potentials, one can use pseudo-scalar potentials and construct a similar model \([13]\). In this case there is no symmetry between the positive- and negative-energy spectra but this is not essential.

5. Summary and discussion

We started with a problem of single-particle quantum mechanics with the Dirac equation for a particle in a given potential. We then considered the HT and QFT versions of the problem and examined how the vacuum energy shifts when an external perturbation is applied. We discussed a situation such that HT and QFT are not equivalent. In HT there are virtual transitions between negative-energy states. The contribution of these transitions, term \( X \) of (6), does not necessarily vanish. If we enforce the Pauli principle in the intermediate states, these transitions do not take place. In QFT there are no such transitions simply because there are no negative energy particles. In case of such discrepancy between HT and QFT, we should choose QFT rather than HT.

Hole theory and QFT are not equivalent for the one-dimensional bag model of ref. 4. This is so for \( m > 0 \) as well as for \( m = 0 \). The perturbation potential \( V(x) \) can be any function of \( x \). Although we assumed a finite bag radius, it can be chosen arbitrarily large. We suspect that inequivalence appears also for the three-dimensional bag model. If we assume a cubic bag rather than a spherical one, the discrepancy between HT and QFT can be easily confirmed. However, we have not confirmed it in more general situations. Example C that we examined suggests that the discrepancy will persist when the sharp boundary of the bag model is replaced by a smooth confining potential. We suspect that the
inequivalence between HT and QFT is a general feature of the type of models that are based on the Dirac equation for a particle interacting with an external potential.

We found a related aspect of QFT that we believe is new. In the type of model that we considered, when an external potential is applied, the vacuum energy can only go down. This is of course well known in second-order perturbation theory, but this is true even when all higher order effects are included. In HT, on the other hand, even the second-order energy shift may not be negative. We also examined a system that has (in addition to the vacuum background) a number of particles (of positive energies).

The models that we have considered in this paper are such that a particle interacts with a given external potential but there is no particle–particle interaction. We, however, suspect that the inequivalence between HT and QFT may be prevalent even when there is a particle–particle interaction (like the Coulomb interaction). In the presence of the particle–particle interaction, with or without an additional external potential, we have more varieties of perturbation processes. In HT it is always possible that one of the particles goes through transitions between negative-energy states, exactly like in the examples that we have examined. Such transitions can lead to inequivalence between HT and QFT in the same way as we have shown. This can be the case even when the particle–particle interaction is the one that is mediated by boson exchanges. In this sense the inequivalence may be quite ubiquitous. If HT and QFT are not always equivalent, some of the calculations done in the framework of HT may have to be re-examined.

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