Renormalization group invariance of quantum mechanics

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Abstract

We propose a framework to renormalize the nonrelativistic quantum mechanics with arbitrary singular interactions. The scattering equation is written to have one or more subtraction in the kernel at a given energy scale. The scattering amplitude is the solution of an nth order derivative equation in respect to the renormalization scale, which is the nonrelativistic counterpart of the Callan–Symanzik formalism. Scaled running potentials for the subtracted equations keep the physics invariant for a sliding subtraction point. An example of a singular potential, that requires more than one subtraction to renormalize the theory is shown.

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1. Introduction

Renormalization and renormalization group techniques have been considered in a wide range of applications after the original works by Wilson [1]. However, as pointed out in Ref. [2], the works of Callan and Symanzik [3] on broken scale invariance in a renormalizable quantum field theory also found important consequences in the theory of critical phenomena. The Callan–Symanzik (CS) equations provide a systematic framework to understand in detail the origin of scaling and universality, as it provides very precise calculational schemes for critical exponents and for universal correlation functions or equations of states. The existence of a scaling limit near a critical point is a direct consequence of the renormalizability of the theory. The CS equations have also shown to be appropriate for dealing with renormalization effects in systems with boundaries, or systems with semi-infinite geometries [4,5].

More recently, the interest in treating singular interactions in Quantum Mechanics, via renormalization procedures, has grown due to the applications of Effective Field Theory (EFT) to Quantum Chromodynamics (QCD) [6–8] and to conventional nuclear physics supplemented by constraints of chiral symmetry [9]. Considering that the Callan–Symanzik
(CS) equations express the invariance of the renormalized theory under renormalization group transformations, it will be appropriate to obtain a formal nonrelativistic extension of such equations by considering the Lippmann–Schwinger scattering equation. This is appropriate, particularly considering the recent applications of the renormalization group equations to quantum mechanics [10–12]. Singular contact interactions have also been considered in a general context [13], as well as in specific treatments of scaling limits and correlations of low-energy observables of three-body systems, in atomic [14] and nuclear physics [15].

The essential physics of the renormalization group in quantum field theory, which controls the sensitivity of the theory to unknown physics at short distances, is expressed by a first order Callan–Symanzik differential equation [16]. In the quantum-mechanical context, as it will be shown, the nonrelativistic counterpart of the Callan–Symanzik (NRCS) equation provides a simple and convenient framework to parametrize in a systematic way the unknown short-distance physics with a minimal number of parameters (the renormalized strengths of the Dirac delta potential and its derivatives). The coefficients/operators that enter in the NRCS equation are finite, independent of the momentum cut-off. Our ignorance on the short distance physics is parametrized, at a given scale \( \mu \), by the renormalized strengths of the singular part of the interaction. This short-range renormalized singular part is added to the corresponding well-known long-range part of the quantum-mechanical interaction. The long distance or low momentum properties of a renormalizable quantum mechanical theory is practically short-distance insensitivity, in the sense that it can be described by a small number of effective parameters, corresponding to the renormalized strengths. In particular, the \( T \)-matrix of a regular potential satisfies the derivative NRCS equation; the physical information carried by the interaction is transferred to the boundary condition at a given scale. The traditional Lippmann–Schwinger equation for a regular potential corresponds to the NRCS equation with the boundary condition given at an infinity scale, where the potential and the \( T \)-matrix are the same.

In this letter, we propose a scheme to renormalize the scattering equation for the \( T \)-matrix, by performing a subtraction in the propagator at a given energy scale. This scheme generalizes the work of Ref. [17]. As an immediate consequence, the scattering amplitude is the solution of a derivative equation in respect to the subtraction scale, which is the nonrelativistic counterpart of the Callan–Symanzik equation. The procedure can be further generalized taking into account higher-order subtractions in the propagator (implying in a higher-order operatorial differential equation), which are necessary when the interaction has singularities stronger than the Dirac delta. The quantum mechanical theory of singular interactions makes sense if it is invariant under the dislocation of the arbitrary renormalization scale. The inhomogeneous term of the subtracted scattering equation runs as the subtraction point moves, constrained by the requirement that the physics described by the theory does not change. With this requirement, the running inhomogeneous term satisfies a renormalization group equation (RGE). In a sense, this renormalization group equation is simply the matching of the theory at scales \( \mu \) and \( \mu + d\mu \), without changing its physical content. The changes in the existing renormalized coefficients are computed by matching the physics just in the vicinity of the boundary between the two scales [6].

This nonrelativistic counterpart of the CS equation for the \( T \)-matrix offers a new view of the potential, as being the boundary condition of a differential operator equation. At the same time, it allows an extension of the scattering equation to singular interactions. This work provides a general treatment for regular plus singular interactions, and it is not concerned with any particular type of regular interaction itself. The regular part of the interaction is particular to the nonrelativistic physical system under study, which ranges from hadronic to atomic scales. The common point in treating different systems is the systematic method to parametrize the unknown short distance physics through a few number of coefficients.

In Section 2, we present the basic steps of the renormalization subtraction procedure, in case where the scattering equation needs only one subtraction to be finite. We also show that such renormalization procedure corresponds to a nonrelativistic counterpart of the Callan–Symanzik (NRCS) equation for the derivative of the \( T \)-matrix in respect to the
subtraction point. In Section 3, we show how to extend the scattering equation with more subtractions in the kernel at a given energy scale. We obtain the corresponding NRCS equation, which is appropriate to treat systems with interactions that have higher-order singularities. We also deduce the differential equation that accounts for the running of the inhomogeneous term of the subtracted $T$-matrix equation with the renormalization scale. In Section 4, the formalism of Section 3 was applied to an example of singular interaction that requires more than one subtraction, in order to obtain the renormalized two-body scattering amplitude. Finally, in Section 5, we present our concluding remarks.

2. One-subtracted $T$-matrix and the Callan–Symanzik equation

The scattering $T$-matrix is not finite for the bare potential containing a Dirac-delta and its derivatives, which imply that regularization and renormalization are required to define the scattering amplitude. The method we suggest consists in constructing regularized and renormalized scattering equations with propagators subtracted at certain scales, which are convenient for introducing the physical inputs. We remind that the method of subtraction has been already used in order to reduce the strength of the kernel of a given integral equation and obtain iterative numerical solutions for the corresponding auxiliary (resolvent) integral equation [18]. However, such an approach has not been used in the context of renormalization of singular interactions and to obtain renormalization group properties of the $T$-matrix.

Let us begin with the operator expression for the scattering equation, the Lippmann–Schwinger equation of the $T$-matrix, at an energy $E$:

$$T(E) = V \left[ 1 + G_0^{(+)}(E) T(E) \right],$$

$$= \left[ 1 + T(E) G_0^{(+)}(E) \right] V,$$ (1)

where the free Green's function, with the appropriate boundary condition, in terms of the free Hamiltonian $H_0$, is $G_0^{(+)}(E) = (E + i\varepsilon - H_0)^{-1}$. The input interaction is given by the $T$-matrix at a given energy scale $-\mu^2$, such that $V$ can be formally written as

$$V = \left[ 1 + T(-\mu^2) G_0(-\mu^2) \right]^{-1} T(-\mu^2).$$ (2)

Introducing the expression (2) in (1), we obtain the $T$-matrix for an arbitrary energy $E$ in terms of the $T$-matrix at a given energy scale $-\mu^2$: $T(E) = T(-\mu^2)$

$$+ T(-\mu^2)(G_0^{(+)}(E) - G_0(-\mu^2))T(E).$$ (3)

We observe that Eq. (3) has the same operatorial form as the original $T$-matrix equation, with the singular interaction $V$ replaced by the $T$-matrix expression at a given energy scale $-\mu^2$, and the original propagator replaced by a propagator which has a subtraction at such energy scale. By defining the renormalized interaction $V^{(1)}(-\mu^2) \equiv T(-\mu^2)$, and a new free Green function

$$G_0^{(+)}(E; -\mu^2) \equiv G_0^{(+)}(E) - G_0(-\mu^2)$$

$$= \left( \mu^2 + E \right) G_0(-\mu^2) G_0^{(+)}(E),$$ (4)

we can write the renormalized $T$-matrix equation with subtracted kernel (3) as

$$T(E) = V^{(1)}(-\mu^2)$$

$$+ V^{(1)}(-\mu^2) G_0^{(+)}(E; -\mu^2) T(E).$$ (5)

In a two-body system, for a potential $V$ as singular as a $\delta$ interaction, Eq. (5) produces finite results once $V^{(1)}(-\mu^2)$ is given. In this case, the momentum representation of $V^{(1)}(-\mu^2)$ is given by the renormalized coupling constant, such that $V^{(1)}(-\mu^2) = \lambda_1(\mu)$.

By taking the limit $E \to -\mu^2$, from Eqs. (3) and (4) we obtain a renormalized equation for the $T$-matrix, which can be written in the form of a nonrelativistic Callan–Symanzik (NRCS) equation:

$$\frac{d}{dE} T(E) \bigg|_{E = -\mu^2} = -T(-\mu^2) G_0^{(+)}(-\mu^2) T(-\mu^2).$$ (6)
The above result implies that, if one subtraction is
eough to render \( T \) finite, one needs to know only the
\( T \)-matrix at a certain energy scale \( -\mu^2 \) and
integrate (6). The NRCS Eq. (6) generalizes the
renormalization group equation that was obtained previously [11] in the context of a two-body system
with a contact interaction.

Considering our starting point, with \( T(E) \) having
no dependence on the scaling parameter, no effect
should be expected from such sliding scale [9,19],
which is true as long as \( T( -\mu^2 ) \) satisfies Eq. (6). In
this way, the quantum mechanical renormalization
scheme is invariant under the renormalization group
transformations. We will come back to this point in
Section 3.1.

3. \( n \)-subtracted \( T \)-matrix and the generalized
Callan–Symanzik equations

In Eq. (3) we should observe that the form factor
modifying the propagator essentially regularizes the
integrand in case the original singular interaction \( V \)
is replaced by \( V^{(1)} \), and the theory is renormalized
once \( V^{(1)} \) is directly connected to an observable.
Further subtractions in the kernel of the equation are
needed when the singularity in the original interaction
\( V \) is stronger than a delta interaction. Within
this perspective, we generalize the subtracted \( T \)-matrix
equation considering \( n \geq 2 \) subtractions in the
kernel at an energy scale \( -\mu^2 \). By taking the one-
subtracted \( T \)-matrix \( (n = 1) \), given by Eq. (5), and
using explicitly Eq. (4) for the propagator, we can obtain the
\( T \)-matrix expression with \( n = 2 \), as follows:

\[
T(E) = V^{(1)}(-\mu^2) + V^{(1)}(-\mu^2)(-\mu^2 - E)
\times \left[ G_0(-\mu^2) G_0^{(+)}(E) - G_0(-\mu^2) \right] T(E)
+ V^{(1)}(-\mu^2)(-\mu^2 - E) G_0(-\mu^2) T(E)
= V^{(2)}(-\mu^2; E)
+ V^{(2)}(-\mu^2; E) G_0^{(+)}(E; -\mu^2) T(E). 
\]  

The last step of the above equation was simplified by
the definitions

\[
V^{(2)}(-\mu^2; E) 
\equiv \left[ 1 - V^{(1)}(-\mu^2)(-\mu^2 - E) G_0(-\mu^2) \right]^{-1}
\times V^{(1)}(-\mu^2), \tag{8}
\]

\[
G_0^{(+)}(E; -\mu^2) 
\equiv \left[ (-\mu^2 - E) G_0(-\mu^2) \right]^{2} G_0^{(+)}(E). \tag{9}
\]

From (8), we observe an explicit dependence on the
energy in the renormalized interaction. In the same
way, we can obtain the \( n \)th order subtracted equation
for the \( T \)-matrix:

\[
T(E) = V^{(n)}(-\mu^2; E) 
+ V^{(n)}(-\mu^2; E) G_0^{(+)}(E; -\mu^2) T(E), \tag{10}
\]

where

\[
V^{(n)}(-\mu^2; E) \equiv \left[ 1 - (-\mu^2 - E)^{n-1} \right]
\times V^{(n-1)}(-\mu^2; E) G_0(-\mu^2) \left[ (-\mu^2 - E)^{n} \right]^{-1}
\times V^{(n-1)}(-\mu^2; E), \tag{11}
\]

\[
G_0^{(+)}(E; -\mu^2) 
\equiv \left[ (-\mu^2 - E) G_0(-\mu^2) \right]^{n} G_0^{(+)}(E). \tag{12}
\]

For example, the renormalization of the \( \delta \)-interaction
is possible with just one subtraction in the
propagator (see Section 2). In general, when the
original interaction is given by a regular potential
supplemented by a \( \delta \)-interaction we can define the
renormalized two-body interaction as [17]

\[
V^{(1)}(-\mu^2) = V_{\text{reg}}^{(1)}(-\mu^2) + \lambda_{\text{reg}}(\mu), \tag{13}
\]

where \( V_{\text{reg}}^{(1)}(-\mu^2) \) can be chosen as the \( T \)-matrix of
the regular potential at the energy \( -\mu^2 \). The value
of \( \lambda_{\text{reg}}(\mu) \) can be adjusted to an observable, as the
scattering length.
The higher-order singularities of the two-body potential are introduced in the inhomogeneous term of the $n$th order subtracted $T$-matrix, such that:

$$V^{(n)}(-\mu^2;E) = \left[1 - (-\mu^2 - E)^{n-1}\right]$$

$$\times \frac{\partial^n V^{(n-1)}(-\mu^2;E)G_0^n(-\mu^2)}{\partial \mu^2}$$

$$\times \frac{\partial V^{(n-1)}(-\mu^2;E)}{\partial E}$$

$$+ V^{(n)}_{\text{sing}}(-\mu^2). \quad (14)$$

$V^{(n)}_{\text{sing}}(-\mu^2)$ is the renormalized strength of the singular part of the interaction, which requires $n$ subtractions in the kernel of (10) to produce a finite $T$-matrix.

3.1. The sliding scale

The coefficients that appears in the scattering amplitude can be traced back to the prescription used to define the renormalized strengths. The central idea of the renormalization group method is to change this prescription [19]. The invariance of the observables under the arbitrary subtraction point, allows one to start at a convenient energy scale $\mu^2$. Such invariance of the renormalized scattering amplitude will result in a definite prescription to modify $V^{(n)}$ in Eq. (10), without altering the predictions of the theory. Thus, the renormalized $T$-matrix is independent on $\mu^2$.

$$\frac{\partial T(E)}{\partial \mu^2} = 0. \quad (18)$$

By using Eqs. (10) and (18), we obtain the renormalization group equation in the form of a NRCS:

$$\frac{\partial V^{(n)}(-\mu^2;E)}{\partial \mu^2}$$

$$\times G_0^{n+1}(-\mu^2)T(-\mu^2) - V^{(n)}(-\mu^2;E)G_0^{n+1}(E; -\mu^2) \frac{\partial G_0^{n+1}(E; -\mu^2)}{\partial \mu^2}$$

$$\times V^{(n)}(-\mu^2;E). \quad (19)$$

This equation expresses the invariance of the renormalized $T$-matrix under dislocation of the subtraction scale $\mu$; however, to keep the theory unchanged as $\mu$ slides, the inhomogeneous term of Eq. (10), $V^{(n)}$, must also vary accordingly. Next, we obtain the differential RGE that expresses the $\mu$ dependence of $V^{(n)}$.
In particular for \( n = 1 \), Eq. (19) is identical to the NRCS Eq. (6). This close our formal presentation of the invariance under renormalization group transformation of quantum mechanics of singular potentials. In the next section we exemplify the application of the proposed framework.

4. Example: renormalized \( T \)-matrix for two-body four-term singular interaction

Let us consider the following four-term singular interaction, after partial-wave decomposition to the s-wave. The bare potential is given by

\[
\langle p|V|q\rangle = \sum_{i,j=0}^{1} \lambda_{ij} p^{2+i}q^{2+j} \left( \lambda_{ij} = \lambda_{ji} \right). \tag{20}
\]

In order to have all the integrals finite with the potential (20), we need at least \( n = 3 \) in Eq. (10). The inhomogeneous term of the scattering equation is given by Eq. (14) with \( n = 3 \). It is constructed by recurrence, such that:

\[
\langle p|V^{(1)}(-p^{2})|q\rangle = \lambda_{\#00},
\]

\[
\langle p|V^{(2)}(-p^{2};k^{2})|q\rangle = \left[ \lambda_{\#00}^{-1} + I_{0} \right]^{-1},
\]

\[
\langle p|V^{(3)}(-p^{2};k^{2})|q\rangle = \bar{\lambda}_{\#00} + \lambda_{\#10}(p^{2} + q^{2}) + \lambda_{\#11} q^{2}, \tag{21}
\]

where \( \bar{\lambda}_{\#00} \equiv \left[ \lambda_{\#00}^{-1} + I_{0} + I_{1} \right]^{-1} \). The units are such that \( \hbar \) and the mass are equal to one. \( I_{i} \) are functions of \( E = k^{2} \) and \( \bar{\mu} \), defined by:

\[
I_{i} = I_{i}(k^{2},\bar{\mu}^{2}) = \frac{2}{\pi} \int_{0}^{\infty} dq \left( \frac{\mu^{2} + k^{2}}{\mu^{2} + q^{2}} \right)^{1+i},
\]

\[
= \left( \frac{\mu^{2} + k^{2}}{2\mu} \right)^{1+2i} \left( i = 0,1 \right). \tag{22}
\]

The s-wave phase-shifts, of the four-term renormalized potential (21), are obtained from the angular momentum projection of the renormalized \( T \)-matrix (10) with \( n = 3 \):

\[
k\cot \delta = -\frac{1}{\bar{\mu}} + \frac{1}{\lambda_{\#00}} - \left[ \left( \bar{\lambda}_{\#00} J_{0} - k^{2} \right) \left( 2\lambda_{\#10} + \lambda_{\#11} k^{2} \right) \right.
\]

\[
+ \lambda_{\#11} J_{1} + \left( \lambda_{\#10}^{2} - \bar{\lambda}_{\#00} \lambda_{\#11} k^{2} \right)
\]

\[
\times \left( \bar{\lambda}_{\#00} J_{0} - J_{0} k^{2} + J_{1} \right)
\]

\[
\times \left[ \bar{\lambda}_{\#00} + \left( 2\lambda_{\#10} + \lambda_{\#11} k^{2} \right) k^{2} \right.
\]

\[
+ \left( \lambda_{\#10}^{2} - \bar{\lambda}_{\#00} \lambda_{\#11} k^{2} \right) \left( J_{0} k^{2} - J_{1} \right) \right]^{-1}, \tag{23}
\]

where

\[
J_{i} = J_{i}(k^{2},\bar{\mu}^{2}) = \frac{2}{\pi} \int_{0}^{\infty} dq \left( \frac{\mu^{2} + k^{2}}{\mu^{2} + q^{2}} \right)^{3}
\]

\[
= \left( \frac{k^{2} + \mu^{2}}{2\mu} \right) \left( 3\mu^{2} \right)^{i} \left( i = 0,1 \right). \tag{24}
\]

The renormalized strengths are found by fitting some observables. In case we have a single constant term for the interaction and \( \lambda_{\#i,j} = 0 \), except for \( \lambda_{\#00} \), we can use the scattering length \( a \) with \( a^{-1} = \bar{\mu} + (\lambda_{\#00})^{-1} \). Thus, the expression for the phase-shift becomes independent of \( \bar{\mu} \). However, when we switch on the other strengths \( \lambda_{ij} \) there is no such simple composition of the strength with \( \bar{\mu} \). From the expression (23) for \( k\cot \delta \), we can anticipate the non-trivial dependence on the scale \( \bar{\mu} \) in the low energy observables. Besides the scattering length, we need other observables to adjust the coupling constants, which can be the effective range and the phase-shift at some specified higher energy. In order to do that, we have also to fix the scale \( \bar{\mu} \) at some value, or better interpret \( \bar{\mu} \) as part of the physical information input.
It is important to observe that the subtraction point can slide without modifying the physical content of the parametrization, if the dependences of the coupling constants on the subtraction point are determined from the solution of the NRCS Eq. (19). The boundary conditions for Eq. (19) are determined by the renormalized coupling constants at the specified value of $\mu$ where the physical inputs are known, given by $V^{(3)}(-\mu^2;k^2)$ in Eq. (21). In the particular case of Eq. (23), the physical inputs are translated to the four parameters $\lambda_{ij}$ ($i(j) = 0, 1$) and $\mu$.

In Fig. 1, we present the s-wave phase shifts results to exemplify our approach. In this example, all units are given in terms of the effective range, such that $r_0 = 1$. The regular potential in momentum space, chosen to generate the data to be fitted by Eq. (23), is given by

$$\langle p|V|q\rangle = \frac{g_s^2}{(2\pi)^3} \left(|p-q|^2 + m_s^2\right)^{-1},$$  \hspace{1cm} \hfill (25)

where $m_s = 6.3176$, $g_s = 26.9544$. In this case, the scattering length is $a = -11.73$. We constraint the scattering length to characterize a short-range interaction, with value of about one order of magnitude larger than $r_0$. There is no other particular meaning to the given value, besides that it leads to a virtual state. The s-wave phase-shift is calculated using this potential and represented in Fig. 1 by the solid line. Next, we try to describe the same phase-shift data using a singular two-term interaction, by adjusting the coupling constants and the subtraction point. For convenience, we choose the dimensionless coupling constants $\lambda_{10}$, $\lambda_{11}$ as being equal to a single value $g$. Thus, three known physical quantities are enough to fix $\lambda_{10}$, $g$ and $\mu$, for which we use the effective range ($r_0 = 1$), the scattering length ($a = -11.73$), and the zero of the phase-shift. The scaling parameter $\mu$ obtained was 2.211, and the constants were $\lambda_{10} = 4.5/\mu^3$ and $\lambda_{11} = 4.5/\mu^3$. As shown in Fig. 1, our fit works up to $E_{\text{lab}} \approx 40 r_0^{-2}$. In nuclear scale (where $r_0 \approx 2.7$ fm), this result corresponds to $\approx 220$ MeV. We have not attempted to do an exhaustive search of parameters to better fit the data, as we just aimed to exemplify the power of the present approach.

5. Concluding remarks

The main purpose of this letter is to present a non-perturbative renormalization procedure to treat singular interactions at any order in nonrelativistic quantum mechanics. The procedure we suggest consists in a subtraction scheme applied to the propagator of the singular scattering equation, which is done at certain energy scale. By following this, we have proved (Section 3) that the subtracted scattering equation is equivalent to a nonrelativistic extension of the Callan–Symanzik equation. The NRCS equations also express the invariance of the theory under renormalization group transformations. This main physical principle is also directly verified in our approach. Such general framework for the nonrelativistic renormalization, and renormalization group analysis, at least from our knowledgement, is pointed out for the first time.

The approach consists in two basic steps: i) regularization of the integrand of the original scattering equation, by using a given subtraction procedure in the propagator, at some energy scale; ii) renormalization is followed by the inclusion of the divergent

![Fig. 1. Two-body s-wave phase shift as a function of the laboratory energy in units of $r_0^{-2}$. The solid line is the calculation with the Yukawa potential (25). The dashed line is the calculation of the phase-shifts with Eq. (23) for the singular potential (20).](image-url)
part of the momentum integration of the propagator in the strengths of the interaction, which is defined by the $T$-matrix and its derivatives at certain energy scale. The subtraction scale can be moved without modifying the calculated observables, which implies that the inhomogeneous term of the subtracted scattering equation runs with the scale.

The method is exemplified for the two-body s-wave scattering, considering the phase-shifts given by a short range Yukawa potential, as the physical data. By using the proposed subtraction method for renormalization, we show how far one can describe the given data by using a singular interaction in second order. Realistic applications of the renormalization approach suggested here for nonrelativistic two and three particle systems, will be considered in more detailed future works.

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