Theory of small aspect ratio waves in deep water

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Abstract

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1. Introduction

The description of the propagation of surface waves in an ideal incompressible fluid is still a classical subject of investigation in mathematical physics as no definite comprehensive answer to the problem has been given yet. In the limit of shallow water, surface gravity waves have been intensively studied and many model equations were introduced by various approaches, with great success. The nonlinear deep water case is more cumbersome and there does not exist today a simple model as universal as the shallow water equations (Korteweg-de Vries or Boussinesq) which would result from an asymptotic limit of the Euler system.

The inherent technical differences between shallow and deep water are mainly due to the fact that the two natural small parameters used for perturbative analysis of the Euler system in shallow water loose their sense in the deep water case (depth $h \to \infty$). Indeed, these parameter are $\epsilon_1 = a/h$, which measures the amplitude $a$ of the perturbation scaled to the fluid depth $h$, and $\epsilon_2 = h^2/\lambda^2$, which measures the depth in units of wavelength $\lambda$.

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Abstract

In the limit of small values of the aspect ratio parameter (or wave steepness) which measures the amplitude of a surface wave in units of its wave-length, a model equation is derived from the Euler system in infinite depth (deep water) without potential flow assumption. The resulting equation is shown to sustain periodic waves which on the one side tend to the proper linear limit at small amplitudes, on the other side possess a threshold amplitude where wave crest peaking is achieved. An explicit expression of the crest angle at wave breaking is found in terms of the wave velocity. By numerical simulations, stable soliton-like solutions (experiencing elastic interactions) propagate in a given velocities range on the edge of which they tend to the peakon solution.

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By perturbative expansion in \( \epsilon_1 \), and fixed finite \( \epsilon_2 \), one obtains the nonlinear shallow water equation [1]. Retaining \( \epsilon_1 \) and \( \epsilon_2 \) (but not their product) leads to different versions of the Boussinesq equations [2], from which the Korteweg-de Vries equation is derived by assuming a small amplitude wave moving in a given direction [3]. All these model equations govern asymptotic dynamics of long wavelength wave profiles.

One way to obtain a small parameter in deep water is to take into account that deep water waves typically result from a superposition of wave components close to a fundamental carrier wave. The small parameter then measures the envelope variations (scaled to the carrier wave). Such nonlinear modulation of wave trains is worked out perturbatively by means of slowly varying envelope approximation (SVEA) which usually leads, in 1+1 dimensions, to the nonlinear Schrödinger model [4,5]. For a full account on modulation of short wave trains on water of intermediate or great depth we refer to [6,7]. The procedure provides the nonlinear dynamics of surface waves as a modulation, the drawback being that the dynamics of the wave profile itself remains unknown.

Based on the theory of analytical functions and perturbation theory, a model for the profile of the free surface wave in water of finite depth, involving the Hilbert transform operator, was derived in [8]. Although this model possesses a well-defined deep water limit, the resulting equation cannot be studied by known techniques to compare it to KdV-like models. Other model equations, built to fit the properties of waves on deep water can be found in [9,10]. Their dispersion relations coincide exactly with that of the water waves on infinite depth but their nonlinear terms are chosen ad hoc to reproduce Stokes waves.

Our purpose here is to study the asymptotic dynamics of the very profile of a surface wave in deep water in the weak nonlinear limit by assuming a dependence on the vertical coordinate close to the linear one. The dispersion relation for a fluid of depth \( h \)

\[
\omega^2 = gh \tanh(\omega h),
\]

leads for long waves on shallow water (parameter \( kh \) small) to the nondispersive relation

\[
\omega = \sqrt{gh} \Rightarrow v_p = \sqrt{gh},
\]

where \( v_p \) is the phase velocity and \( v_g \) the group velocity. However, waves on deep water (parameter \( kh \rightarrow \infty \)) are dispersive as from (1.1)

\[
\omega = \sqrt{gh} \Rightarrow v_p = \sqrt{\frac{g}{k}}, \quad v_g = \frac{1}{2} \sqrt{\frac{g}{k}}.
\]

This deep water dispersion relation will be one of our main guides in the process of finding a limit model whose linear limit is constrained by (1.3).

Our approach follows the method of Green and Naghdi [11,12] for surface waves in shallow water which assumes an anzatz for the dependence of the velocity components on the vertical dimension \( z \). This anzatz does not produce an exact solution of the full Euler system and the game consists in replacing one of the equations with its integrated expression. This comes actually to making an average over the depth, which can be performed with different weights. Although weight is not determinant in the shallow water case [13], we shall see that its choice is prescribed by a consistency requirement within the linear limit.

A limit model is then obtained by defining a small parameter which measures the amplitude of a surface wave in units of its wave-length, we call it the aspect ratio parameter (ARP), it is also referred to as the wave steepness.

Our approach combines the asymptotic analysis à la Whitham with the already standard method of multiple scales [14–18], it will be shown to lead to the following model

\[
\frac{\eta_t - \eta_{xx}}{\eta} = \frac{1}{2} \frac{\eta_{xxt} + \frac{3}{2} \eta_{xxx} + \eta_{x}}{\eta} = \frac{5}{2} \eta_1 \eta_{xx} + \frac{1}{2} \eta_2 \eta_{xxx},
\]

for the dimensionless deformation \( \eta(x, t) \) of the deep water free surface.

The paper is organized as follows. In Section 2, we introduce the Euler equations, their nondimensional version and the anzatz which, together with a convenient average, enables to reduce the initial three-dimensional problem
to a two-dimensional one. The resulting model, Eq. (2.25) below, is in fact the deep water version of the celebrated Green-Naghdi system for surface waves in shallow water [11,12].

In Section 3, the nonlinear dispersive model (1.4) that governs propagation of waves of small ARP in deep water is derived by the method of multiple scales and perturbative expansion. Expressed in physical units, the model appears with \( k \)-dependent coefficients, reminiscent of what occurs in the SVEA approach of deep water modulation.

In Section 4, analytical and numerical analysis of the progressive periodic waves are performed. Numerical simulations show the existence of periodic waves that tend to peak as amplitude grows. The crest angle at the peak is calculated and we obtain an explicit expression in terms of the wave velocity. Moreover, numerical simulations show also the existence of soliton-like solutions (though we cannot provide analytic expression) that tend, at the edge of the allowed velocity range, to the peakon solution (an explicit analytical expression of which is given). These soliton seem to undergo elastic insteractions, though we have not been able to obtain energy conservation out of (1.4).

2. The model equations in deep water

2.1. General settings

Let us consider the Euler equations in physical dimensions where the particles of the fluid are identified in a fixed rectangular Cartesian system of center \( O \) and axes \( (x, y, z) \) with \( Oz \) the upward vertical direction. We assume translational symmetry in \( y \) and thus consider a sheet of fluid in the \( xz \) plane. The velocity of the fluid in this plane is the vector \( (U, W) \) where \( U(x, z, t) \) is the horizontal component and \( W(x, z, t) \) the vertical one. This fluid sheet is moving on a bottom at \( z = -\infty \) and its free surface is \( z = \eta(x, t) \).

The continuity equations and the Newton equations in the flow domain read
\[
U_x + W_z = 0, \\
\rho \dot{U} = -P_x, \\
\rho \dot{W} = -P_z - g\rho,
\]
where \( P(x, z, t) \) denotes the pressure, \( \rho \) the uniform density and \( g \) is the gravity. Subindices mean partial derivatives and overdot means material derivative defined as usual by
\[
\dot{F} = Ft + UF_x + WF_z.
\]
The boundary conditions at \( z = -\infty \) simply state that the velocity component vanish \((U \to 0 \text{ and } W \to 0)\), while boundary conditions at the free surface \( z = \eta(x, t) \) state that \( P \) is the atmospheric pressure \( P_a \) and that the total derivative of the surface equation \( z - \eta = 0 \) vanishes, namely
\[
z = \eta : \quad \eta_t + U\eta_x - W = 0.
\]

We are interested here in finding the evolution equations for the free surface by studying the nonlinear deformation of a particular linear wave profile with given arbitrary wave number \( k \).

2.2. Dimensionless Euler system

The linear progressive monochromatic solution of the linear limit of the above Euler system reads [19]
\[
U_{\text{linear}} = U_0 \cos(kx - \omega t) e^{kz}, \quad \omega^2 = gk,
\]
where the dispersion relation is indeed of the deep-water class (1.3).
Thus, given the wave number $k$, we may scale the original space variables $x$ and $z$ with $k$, the time variable with $\sqrt{kg}$, the velocity components $U$ and $W$ with $\sqrt{kg}$, and the pressure $P$ with $k/\rho g$. For convenience we keep the same notations for the adimensional variables and the Euler equations for $z \in [-\infty, \eta]$ then become

\begin{align}
U_x + W_z &= 0, \\
\dot{U} &= -P_z, \quad (2.7) \\
\dot{W} &= -P_z - 1, \quad (2.8)
\end{align}

with the boundary conditions

\begin{align}
z \to -\infty: \quad &U = W = 0, \quad (2.10) \\
\eta: \quad &P = P_0, \quad (2.11) \\
\eta: \quad &\eta_t = W - U \eta_x, \quad (2.12)
\end{align}

where $P_0 = P_0 \sqrt{kg}$. Note that, as $z$ is scaled with $k$, the dimensionless surface profile $\eta$ is also scaled with $k$.

### 2.3. Vertical velocity profile

Inspired thus by (2.6), we assume an exponential vertical dependence of the horizontal velocity component and seek an approximate solution of the Euler system by starting with the anzatzz$U(x, z, t) = u(x, t)e^z$.

\begin{align}
\dot{U} &= ut e^z, \quad (2.15) \\
\dot{W} &= -uxt e^z - u_{xx} e^{2z} + u^2 x e^{2z}, \quad (2.16)
\end{align}

which allows to calculate explicitly the pressure $P$ from (2.9) and boundary condition (2.11) as

\begin{align}
P - P_0 = (\eta - z) - u_{xx}(e^z - e^\eta) + \frac{1}{2}(u^2_x - uu_{xx})(e^{2\eta} - e^{2z}). \quad (2.17)
\end{align}

This pressure diverges as $z \to -\infty$ in an Archimedean way as it must.

The boundary condition (2.12) finally provides the evolution of $\eta$ as

\begin{align}
\eta_t + (u e^\eta)_{x} &= 0. \quad (2.18)
\end{align}

To that point the anzatz (2.13), the formula (2.14), the expression (2.17) of $P$ and the evolution (2.18) of the free surface $\eta$ satisfy exactly the differential Eqs. (2.7) and (2.9) together with the whole boundary conditions (2.10), (2.11) and (2.12). The remaining equation to take into account is then the Newton’s law (2.8) which is of course not satisfied globally by the above expressions of $W$ and $P$. 

2.4. Averaging Newton’s law

The Newton’s law (2.8) is now taken into account through an average over the full depth with the weight \( e^{\alpha z} \) which regularizes the diverging Archimedean term in the expression of the pressure (2.17). This is explicitly

\[
\int_{-\infty}^{\eta} dz \, e^{\alpha z} (U + P_z) = 0.
\]

which eventually furnishes with use of (2.15) and (2.17)

\[
\frac{\alpha + 1}{a + 1} u_{\eta} = \frac{1}{a + 1} \left( \frac{\alpha + 2}{a + 1} u + \frac{\alpha + 3}{a + 2} \left[ uu_{xx} - u_x^2 \right] - \frac{\alpha}{a} \right).
\]

It is a model-dependent system where the solutions now depend on the external parameter \( \alpha \).

The value of parameter \( \alpha \) is fixed by demanding that the linear limit of the system (2.18) (2.20) possesses solutions with phase and group velocities given by (1.3). In dimensionless units, these velocities are then required to be

\[
v_p = 1, \quad v_g = \frac{1}{2}.
\]

The linear limit of the system (2.18) (2.20), after having eliminated \( \eta \), can be written

\[
\frac{1}{2} u_{tt} + (\alpha + 1) u_{xx} - uu_{tt} = 0,
\]

which possesses the solution

\[
u = u_0 \cos(qx - \omega t), \quad \omega^2(q) = \frac{(\alpha + 1)q^2}{a + q^2}.
\]

Requiring relation (2.21) for the above dispersion law \( \omega(q) \) comes to require the relations

\[
v_p = \omega(q) \bigg|_{q=1} = 1, \quad v_g = \frac{\partial \omega}{\partial q} \bigg|_{q=1} = \frac{1}{2}.
\]

The first of these equations is verified for any \( \alpha \) while the second holds if and only if \( \alpha = 1 \).

As a result the basic system of equation is (2.18) and (2.20) written with \( a = 1 \), namely

\[
\frac{1}{2} u_{tt} + \beta u_{\eta} \left( u^2 - \frac{1}{2} uu_{xx} \right) = 0,
\]

\[
\eta_t + (u \eta)_{\eta} = 0.
\]

This system for the couple of variables \( u(x, t) \) and \( \eta(x, t) \) is the net result of assuming the \( z \)-dependence (2.13) in the Euler equations and of averaging Newton’s law on the depth \((-\infty, \eta)\). As displayed above, a crucial step is the use of the precise weight \( e^{\alpha z} \), while it can be proved from [13] that weighting the average is not determinant in the shallow water case.

3. Small aspect ratio waves in deep water

3.1. Generalities

The small parameter of the perturbative analysis of the system (2.25) is, in dimensionless units, the maximum amplitude \( \epsilon \) of the free surface deformation \( \eta(x, t) \). So we define

\[
\eta(x, t) = \epsilon H(x, t),
\]

where

\[
H(x, t) = \frac{\partial \eta}{\partial t}.
\]
and it is instructive to understand the physical meaning of $\epsilon$ by returning to the physical units for which

$$\epsilon = ak,$$  

where $a$ is the wave profile maximum amplitude and $k$ the chosen wave number. Thus, $\epsilon$ is the parameter that measures the ratio of wave height to wave length (ARP) and a small $\epsilon$ means a flat deformation of the surface without reference to the absolute amplitude or to the depth.

The dimensionless linear dispersion relation ($\omega^2 = 1$) undergoes deviations due to nonlinearity (Stokes’ hypothesis) which can be taken into account through an expansion of $\omega^2$ in powers of $\epsilon$. Therefore, the phase $x - \omega t$ expands also in powers of $\epsilon$ which leads to a representation of $H(x, t)$ as the function $H(y, \tau, \nu, \ldots)$ of the new variables $y, \tau, \nu, \ldots$, defined by

$$y = x - t, \quad \tau = \epsilon t, \quad \nu = \epsilon^2 t, \ldots$$  

(3.3)

For those variables we have the following derivation rules

$$\partial_x = \partial_y, \quad \partial_t = -\partial_y + \epsilon \partial_\tau + \epsilon^2 \partial_\nu + \cdots$$  

(3.4)

which are now explicitly applied step by step to our system (2.25).

3.2. Asymptotic expansion

Since $\exp(\epsilon H) = 1 + \epsilon H + \mathcal{O}(\epsilon^2)$ we can consider system (2.25) at order $\epsilon^0$, next at orders $\epsilon^0$ and $\epsilon$ and so on. In terms of $H, y$ and $\tau$ we get

$$(-\partial_y + \epsilon \partial_\tau + \mathcal{O}(\epsilon^2)) [\epsilon H + [\mu(1 + \epsilon H + \mathcal{O}(\epsilon^2))]_y = 0$$  

(3.5)

This equation gives $u$ in term of derivatives and integrations of $H$

$$u = \epsilon H - \epsilon^2 \left( H^2 + \int_{-\infty}^{y} H \, dy' \right) + \mathcal{O}(\epsilon^3)$$  

(3.6)

where it is not assumed that $u$ or $H$ (and derivatives) vanish as $y \to -\infty$. Instead the above equation readily gives the useful relation

$$u(-\infty, \tau) = \epsilon H(-\infty, \tau)[1 - \epsilon H(-\infty, \tau) + \mathcal{O}(\epsilon^2)].$$  

(3.7)

Now Eq. (2.25) is more conveniently written in the form

$$\frac{1}{2} A + \frac{1}{3} B - C - \frac{1}{2} D = 0$$  

(3.8)

with $A, B, C$ and $D$ given by

$$A = (u \mu e^{2\nu})_y, \quad B = (u u_{x\alpha} - (u_x)^2 e^{2\nu})_y, \quad C = (e^\nu)_y, \quad D = u \mu e^{2\nu}.$$  

(3.9)

Using the operator expansion

$$\partial^2_{y\nu} = -\partial^2_{y\nu} + \epsilon \partial^2_{y\tau} + \mathcal{O}(\epsilon^2)$$  

(3.10)

we obtain

$$A = \frac{\partial}{\partial y} \left( 1 + 2\epsilon H + \mathcal{O}(\epsilon^2) \right) \times (-\partial^2_{y\nu} + \epsilon \partial^2_{y\tau} + \mathcal{O}(\epsilon^2)) \left( \epsilon H - \epsilon^2 H^2 - \epsilon^2 \int_{-\infty}^{y} H \, dy' + \mathcal{O}(\epsilon^3) \right).$$
which at orders $\epsilon$ and $\epsilon^2$ provides
\begin{equation}
A = -\epsilon H_{yyy} + \epsilon^2 (\epsilon H_y H_{yy} + 2 H_{yxy}) + \mathcal{O}(\epsilon^3),
\end{equation}
(3.11)
Coming now to the computation of $B$ we may write
\begin{equation}
B = \frac{\partial}{\partial y} \left\{ \left[ 1 + 3\epsilon H + \mathcal{O}(\epsilon^2) \right] \times \left[ \left( \epsilon H - \epsilon^2 H^2 + \epsilon^2 \int_{-\infty}^y H_x dy' + \mathcal{O}(\epsilon^3) \right) \times \left( \epsilon H_y + \mathcal{O}(\epsilon^2) - \epsilon^2 H^2 + \mathcal{O}(\epsilon^3) \right) \right] \right\}
\end{equation}
which at order $\epsilon^2$ provides
\begin{equation}
B = \epsilon^2 (\epsilon H_{yxy} - H_{yy}) + \mathcal{O}(\epsilon^3).
\end{equation}
(3.12)
Finally $C$ and $D$ at order $\epsilon^2$ are given by
\begin{equation}
C = (\epsilon H_y)\eta = \epsilon H_y + \epsilon^2 H \eta_x + \mathcal{O}(\epsilon^3),
\end{equation}
(3.13)
\begin{equation}
D = [-\eta_t + \eta_x + \mathcal{O}(\epsilon^2)](\epsilon H - \epsilon^2 (H^2 + \int_{-\infty}^y H_x dy') + \mathcal{O}(\epsilon^3)) \times [1 + 2\epsilon H + \mathcal{O}(\epsilon^2)]
\end{equation}
(3.14)
We substitute $A, B, C$ and $D$ in (3.8), keep the terms in $\epsilon^0$ and $\epsilon^1$ to obtain eventually
\begin{equation}
-H_y - H_{yyy} + \epsilon \left( \frac{10}{3} H_x H_{yy} + \frac{2}{3} H H_{yxy} - 2 H_{yxx} + 2 H_{xxx} - 2 H \right) = 0
\end{equation}
(3.15)
or in the original (dimensionless) $x$ and $t$ variables
\begin{equation}
H_t + \frac{1}{2} H_x - H_{xxt} - \frac{1}{2} H_{xxx} + \epsilon H H_x = \epsilon \left( \frac{5}{3} H_x H_{xx} + \frac{1}{3} H H_{xxx} \right).
\end{equation}
(3.16)
In terms of the free surface $\eta = \epsilon H$ the above equation reads
\begin{equation}
\eta_t - \frac{1}{3} \eta_{xxx} - \frac{3}{2} \eta_x + \eta_{tt} = \frac{5}{3} \eta_x \eta_{xx} + \frac{1}{3} \eta \eta_{xxx},
\end{equation}
(3.17)
as announced in the introduction. This is the equation which describes asymptotic nonlinear and dispersive evolution of small ARP waves in deep water.

3.3. Discussion

We note first that the evolution (3.17) of the free surface, written in physical dimensions, namely by $\eta \rightarrow k \eta$, $x \rightarrow k x$ and $t \rightarrow k \sqrt{g/\kappa}$, reads
\begin{equation}
\eta_t - \frac{1}{27} \eta_{xxx} - \frac{1}{27} \sqrt{g/k} \eta_{xxt} + \frac{3}{27} \sqrt{g/k} \eta_x + \sqrt{g/k} \eta_{tt} = \frac{1}{3} \sqrt{g/k} \left( 5 \eta_x \eta_{xx} + \eta \eta_{xxx} \right).
\end{equation}
(3.18)
It is therefore a $k$-dependent equation that describes the nonlinear deformations of the wave of given $k$.
Next, by defining
\begin{equation}
F(x, t) = \frac{1}{2} \frac{H(x, t)}{H} + \frac{1}{2}
\end{equation}
(3.19)
we transform the Eq. (3.17) into
\[ F_t - F_{xxt} + 3FF_x = 5F_xF_{xx} + FF_{xxx}. \] (3.20)

The equations belonging to the one-parameter family
\[ F_t - F_{xxt} + (b + 1)FF_x = bF_xF_{xx} + FF_{xxx}, \] (3.21)
are completely integrable only in the two cases \( b = 2 \) (Camassa-Holm equation) [20] and \( b = 3 \) (Degasperis-Procesi equation) [21–23]. Our Eq. (3.20) does not enter this class and we cannot tell more about integrability.

Finally, from the above perturbative expansion we also have, according to (3.6) the same equation as (3.17) for the variable \( u(x, t) \). It can be written as the conservation law
\[ \partial_t \left( u - u_{xx} \right) = \partial_x \left( \frac{1}{2} uu_{xx} - \frac{1}{2} u - \frac{1}{2} u^2 + \frac{1}{3} u^3 + \frac{1}{3} uu_{xx} \right). \] (3.22)

4. Progressive waves

4.1. Periodic progressive waves

The waves of constant profile propagating at velocity \( c \) are the solutions of (3.17) for \( \eta(x, t) = \eta(\xi) \) in the comoving frame \( \xi = x - ct \), which leads to the ordinary differential equation
\[ \left( \frac{3}{2} - c \right) \eta_{\xi} - \left( \frac{1}{2} - c \right) \eta_{\xi\xi\xi} + \eta_{\xi\xi} = \frac{5}{3} \eta_{\xi\xi\xi} + \frac{1}{3} \eta_{\xi\xi\xi}. \] (4.1)

Integrated once this equation becomes
\[ \left( \frac{3}{2} - c \right) \eta - \left( \frac{1}{2} - c \right) \eta_{\xi\xi} + \frac{1}{3} \eta_{\xi\xi\xi} - \frac{2}{3} \eta_{\xi\xi\xi} - \frac{1}{3} \eta_{\xi\xi\xi} = A, \] (4.2)

where \( A \) is an integration constant to be determined from the chosen boundary conditions.

To study the periodic solutions, we integrate (4.1) numerically under the initial values
\[ \eta(0) = 0, \quad \eta_{\xi}(0) = 0, \quad \eta_{\xi\xi}(0) = \alpha, \] (4.3)
and obtain a typical set of profiles displayed in Fig. 1 for different values of the parameter \( \alpha \). Such a simulation shows the wave crest peaking, together with wavelength variations, as wave amplitude increases. We have checked that low amplitude profiles (obtained for small values of \( \alpha \)) do tend to the solution \( \alpha(1 - \cos(\xi)) \) of (4.1) linearized and with \( c = 1 \).

The value of the maximum amplitude \( \eta_s \) reached at wave peaking can be inferred from the Eq. (4.2), it is the value for which the coefficient of the second derivative vanishes, namely
\[ \eta_s = 3 \left( c - \frac{1}{2} \right). \] (4.4)

We have found by numerical check that this occurs for the initial data (4.3) with the threshold \( \alpha_s \) given by
\[ \alpha_s = \frac{3}{10} \left( \frac{5}{2} - c \right). \] (4.5)

At the above threshold values, the crest angle can be evaluated as follows.
Fig. 1. Profiles of the dimensionless surface deformation $\eta$ solution of (4.1) with (4.3) for $c = 1$ and $\alpha$ taking the values 0.2, 0.3, 0.4, 0.446, 0.4499 and 0.45, from bottom to top.

4.2. Crest angle

First the integration constant $A$ of (4.2) is evaluated at $\xi = 0$ by means of (4.3) and (4.5), we obtain

$$A = \frac{3}{10} \left( c - \frac{1}{2} \right) \left( -\frac{5}{2} c + 1 \right).$$

(4.6)

Next we evaluate Eq. (4.1) at the limit $\xi \to \xi^+_n$ (limit to the left $\xi^-_n$ or to the right $\xi^+_n$), where $\xi^+_n$ is the value for which the threshold is reached, namely $\eta(\xi^+_n) = \eta_0$. We obtain this way the limit of the second derivative by

$$\lim_{\xi \to \xi^+_n} (\eta^{\prime\prime}) = \frac{6}{5} v.$$  

(4.7)

To get the limits of the first derivative we evaluate now Eq. (4.2) at $\xi = \xi^+_n$. With help of the above value of $A$ we obtain

$$\lim_{\xi \to \xi^+_n} (\eta^{\prime}) = \frac{9}{2} \left( c - \frac{1}{2} \right) \left( -\frac{6}{5} c + 1 \right),$$

(4.8)

and thus the limit of $\eta_0$ in $\xi^+_n$ can be either positive or negative. It is then clear that a bounded periodic solution like the one displayed on Fig. 1 implies the following solution (remember $c > 1/2$)
Fig. 2. Values of the crest angle, in degrees, at wave peaking for velocities $c$ in the range $[0.6, 1.2]$.

\[
\lim_{\xi \to \xi_{\ell}} (\theta_{\ell}) = \left[ \frac{9}{4} \left( c - \frac{1}{2} \right) \left( \frac{6}{2} c + 1 \right) \right]^{1/2}. \quad (4.9)
\]

Last, from the above value of the slope at $\xi_{\ell}$, the crest angle at wave peaking reads

\[
\theta = 2 \arctan \left\{ \left[ \frac{9}{4} \left( c - \frac{1}{2} \right) \left( \frac{6}{2} c + 1 \right) \right]^{-1/2} \right\}, \quad (4.10)
\]

and the graph of this expression is plotted on the Fig. 2. At $c = 1$ the angle is $\theta = 65.4^\circ$.

It is worth remarking that the analytic expression of the crest limit angle, which we have checked on numerical simulations, actually results in a part from the phenomenological expression (4.5) obtained by observations of numerical simulations of Eq. (4.1) with boundary data (4.3).

4.3. Peakon solution

The coefficients of the ODE (4.1) show that periodic solutions hold if $c > 1/2$ and a threshold is reached at $c = 5/2$ for which waves of vanishing amplitudes are already singular. At this value, the model supports the peakon solution

\[
\eta(\xi) = 6 \exp[-|\xi|/\sqrt{2}], \quad \xi = x - \frac{5}{2} t. \quad (4.11)
\]

Remarkably enough, this peakon also appears as the limit of the solitary wave solution described below when its velocity reaches the value $5/2$. 
4.4. Soliton-like waves

Numerical simulations have revealed the existence of stable soliton-like excitations that travel without deformation at constant speed in the range \([3/2, 5/2]\). These are performed by observing that

\[
\eta(s) = \eta_0 \cosh(\kappa(x - \frac{3}{2}vt - x_0)), \quad \eta_0 = 3 \left( \frac{v - \frac{3}{2}}{2v - \frac{1}{2}} \right), \quad \kappa = \frac{1}{2} \sqrt{\frac{2v - \frac{3}{2}}{2v - 1}},
\]

(4.12)

is an approximate solitary wave solution for (1.4) according to

\[
\eta_{xx}^{(n)} - \frac{\eta_{tt}^{(n)}}{t^2} - \frac{3}{2} \eta_{tt}^{(n)} + \frac{3}{2} \eta^{(n)} + \eta^{(n)} \eta_{xx}^{(n)} - \frac{5}{3} (\eta_{xx}^{(n)} + \frac{1}{3} \eta^{(n)} \eta_{xx}^{(n)}) = O(\kappa^3 \eta_0^2).
\]

(4.13)
Hence defining
\[ v = \frac{3}{2} + \delta \Rightarrow \begin{cases} \eta_0 = 3\delta, & \kappa^2 = \frac{\delta}{1 + \frac{3}{2}}. \end{cases} \] (4.14)

expression (4.12) tends to the exact solution for \( \delta \to 0 \) (for which the amplitude \( \eta_0 \) vanishes).

We have performed a series of numerical simulations of (1.4) by first writing it in the frame moving at velocity \( v \), namely
\[ \eta_t + \left( \frac{3}{2} - v \right) \eta_x + \left( v - \frac{1}{2} \right) \eta_{xxx} + \eta_{xx} = \frac{5}{2} \eta_x \eta_{xx} + \frac{1}{3} \eta_{xxx}. \] (4.15)

Fig. 4. Solitary wave interaction in the rest frame of the soliton at velocity 1.8, the other soliton having velocity 2. The figure displays \( \eta(x, t) \) solution of (4.15), plotted as function of \( x \) for different times.
and then by using the following initial-boundary value problem on $x \in [0, L]$ 

$$
\eta(x, 0) = \eta^{\prime}(x, 0), \quad \eta(0, t) = 0, \quad \eta(L, t) = 0, \quad \eta_x(L, t) = 0, 
$$

with an initial position $x_0 = 2L/3$ and a velocity

$$
v = \frac{3}{2} + \delta, \quad \delta \in [0, 1].
$$

The method uses a standard second order implicit finite difference scheme with a mesh grid dimension 0.1.

A few typical examples are displayed in Fig. 3 where we compare the solitary wave to the initial condition. As expected, when $\delta \ll 1$ (i.e. $v \sim 1.5$) the solitary wave resembles much the initial condition while for larger $\delta$, a pulse reshaping occurs by means of emission of waves (at phase velocity 1). After reshaping, the solution is remarkably stable. When $\delta \to 1$ (i.e. $v \to 5/2$), the solitary wave tends to the peakon solution (4.11).

The found solitary wave solutions behave like solitons do in an integrable system. Indeed, after reshaping, a two-soliton interaction effects only the position of the pulses and does not alter the individual shapes. An example is displayed in Fig. 4

## Conclusion and comments

The assumed exponential vertical velocity profile has led to a simple model for surface waves on deep water. The derivation is rigorously made by perturbative analysis assuming waves with small aspect ratio parameter, which constitutes a novel approach of the problem.

The resulting model (3.17) sustains periodic waves which tend to peak as their amplitude increases and reaches a threshold amplitude $\eta_s$ given by (4.4). At this value, the crest angle can be explicitly computed in terms of the velocity by (4.10). However, in the region of the peak, higher order derivatives diverge and the perturbative expansion does not hold anymore. Hence, the precise value of the crest angle should be considered with care and the peaking only gives an indication of a behavior due to nonlinearity.

We have not been able to obtain an Hamiltonian formulation of Eq. (3.17), although the numerical simulations displayed in Fig. 4 seem to indicate an energy conservation, and although the equation is of the form of integrable systems as the Camassa-Holm or the Degasperis-Procesi equations, see e.g. [22].

As mentionned in the introduction, it is necessary to release the potential flow approach which is not consistent with the anzatz (2.13). Indeed, assuming the existence of a potential flow $\phi$ by

$$(U, W) = \nabla \phi, \quad (4.18)$$

the expressions (2.13) and (2.14), together with (2.7), readily provide

$$\phi(x, z, t) = \psi(x, t)e^z, \quad \psi_{zz} + \psi = 0. \quad (4.19)$$

Although one may obtain nontrivial time-dependence from the Euler system in this particular context, the above linear $z$-dependence of $\psi(x, t)$ cannot catch nonlinear deformations in a consistent way.

Last but not least, it is remarkable that Eq. (3.18) possesses $k$-dependent coefficients, while $k$-dependent coefficients are common in the context of modulation of wave trains (as in nonlinear Schrödinger, modified NLS and Davey-Stewartson). As given by the Eq. (4.1) for permanent profile solutions, the only linear periodic solution allowed by the model is obtained for $c = 1$, that is for the very wavenumber chosen for the anzatz (2.13). However nonlinearity allows for propagation of waves of different velocities that must be understood as deformations of the linear profile at $c = 1$. 


References